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Thermodynamic uncertainty relation for open quantum systems

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Related article:
Thermodynamic Uncertainty Relation for Open Quantum Systems, arXiv:2003.08557v2

Thermodynamic uncertainty relation (TUR)

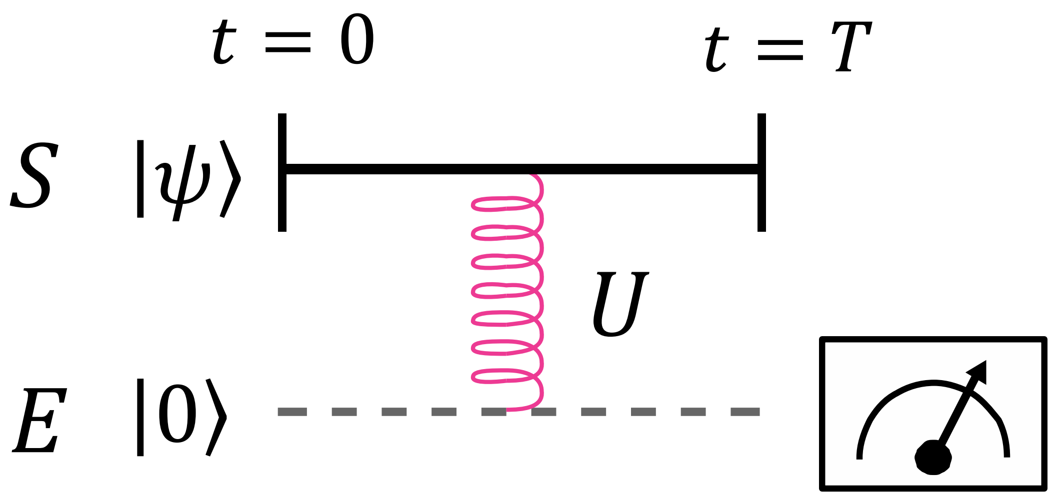
- Relation between fluctuation and entropy production [Barato & Seifert, PRL, 2015]

$$\frac{\mathrm{Var}[\phi]}{\langle\phi\rangle^2} \geq \frac{2}{\sigma}$$

where σ is entropy production.

- Recently, quantum TURs have been studied
  - [Erker *et al*., PRX, 2017], [Brandner *et al*., PRL, 2018], [Carollo *et al*., PRL, 2018], [Liu *et al*., PRE, 2019], [Guarnieri *et al*., PRR, 2019], [Saryal *et al*., PRE, 2019], etc
- Still, quantum TURs are in a very early stage
  - Many studies obtained case-by-case bounds
- I will present a quantum TUR valid for general open quantum dynamics

TUR in open quantum systems



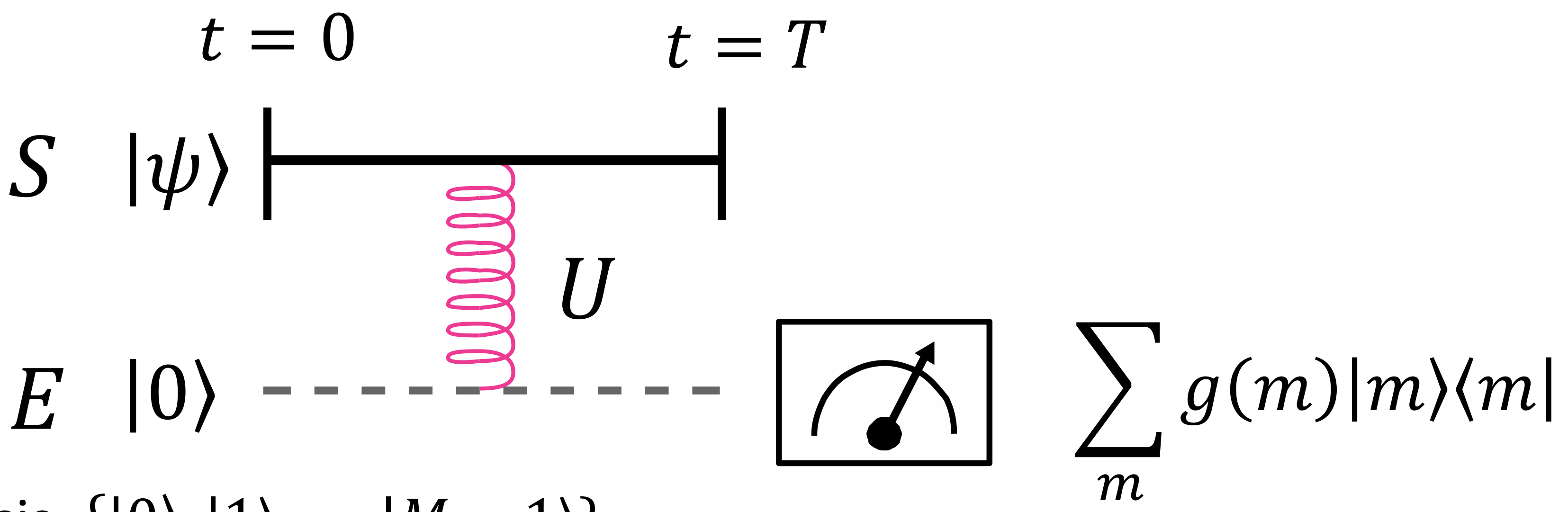
$$\sum_m g(m)|m\rangle\langle m|$$

Environment basis: $\{|0\rangle, |1\rangle, \dots, |M-1\rangle\}$

$$|\Psi(T)\rangle = U|\psi\rangle \otimes |0\rangle = \sum_{m=0}^{M-1} V_m|\psi\rangle \otimes |m\rangle, \qquad V_m \equiv \langle m|U|0\rangle$$

$$\rho(T) = \mathrm{Tr}_E[|\Psi(T)\rangle\langle\Psi(T)|] = \sum_{m=0}^{M-1} V_m \rho V_m^\dagger$$

TUR in open quantum systems



Environment basis: $\{|0\rangle, |1\rangle, \dots, |M-1\rangle\}$

- We assume that

$$g(0) = 0$$

- As long as this condition is met, $g(m)$ can return any real number
- The initial state of E was assumed to be $|0\rangle$. Therefore, when the state of the environment after the interaction is $|0\rangle$, the environment remains unchanged before and after the interaction.

TUR in open quantum systems

- Then we find the following bound for the coefficient of variation of $g(m)$:

$$\frac{\mathrm{Var}[g(m)]}{\langle g(m)\rangle^2} \geq \frac{1}{\Xi}$$

$$\Xi = \mathrm{Tr}_S\left[\left(V_0^\dagger V_0\right)^{-1}\rho\right] - 1 \qquad V_0 \equiv \langle 0|U|0\rangle$$

- Ξ corresponds to the dynamical activity in classical Markov processes
- This relation holds for
  - any open quantum systems as long as $V_0^\dagger V_0 > 0$
  - any observable $g(m)$ with $g(0) = 0$
  - any initial density operator ρ in S

Application: continuous measurement

- Consider a Lindblad equation defined by

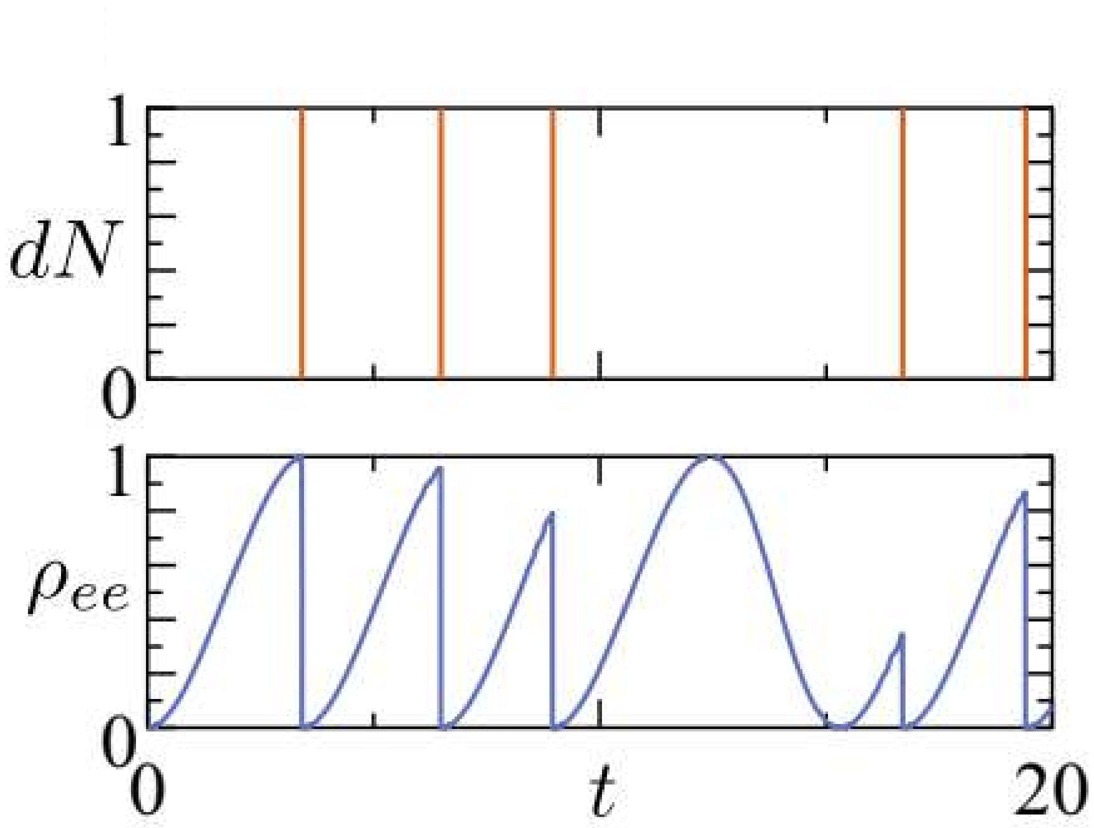
$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_c \left[L_c \rho L_c^\dagger - \frac{1}{2}\{L_c^\dagger L_c \rho + \rho L_c^\dagger L_c\} \right]$$

where L_c is a jump operator.

- The Lindblad equation renders the dynamics when we do *not* measure the environment.
- On measuring the environment, the Lindblad equation is unraveled to yield a stochastic dynamics conditioned on a measurement record
- Stochastic trajectory is described by a stochastic Schrödinger equation

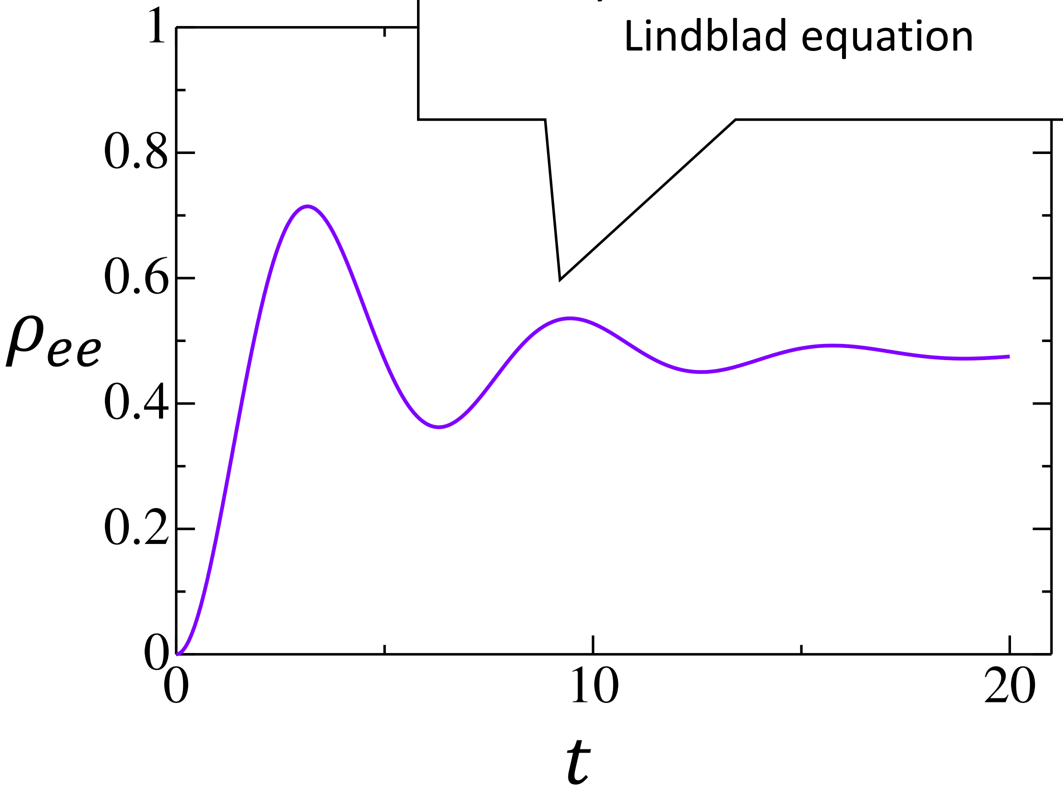
Quantum trajectory

$$d\rho = -i[H,\rho]dt + \sum_c \left[\rho\, Tr\left[L_c \rho L_c^\dagger\right] - \frac{1}{2}\{L_c^\dagger L_c, \rho\}\right] + \sum_c \left[\frac{L_c \rho L_c^\dagger}{Tr\left[L_c \rho L_c^\dagger\right]} - \rho\right] dN$$



$\rho_{ee}(t) \equiv \langle e|\rho(t)|e\rangle$

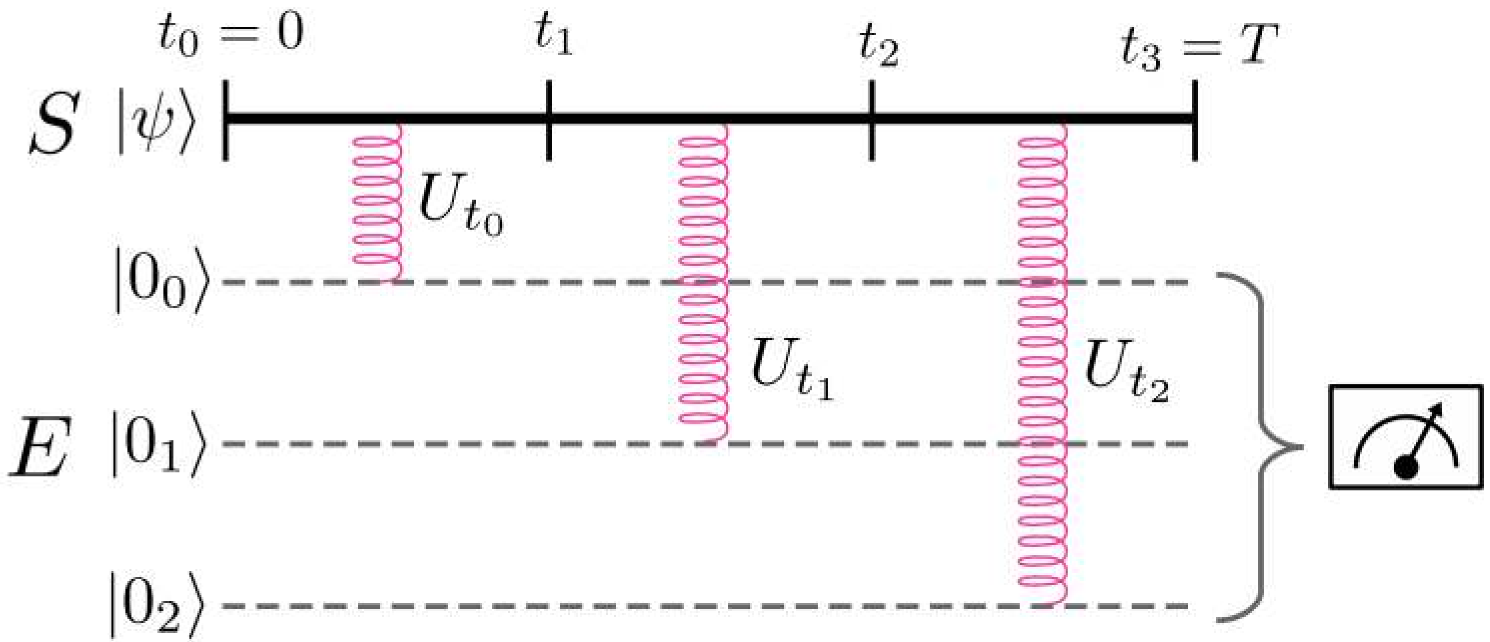
Taking average w.r.t. measurement records



This dynamics is a solution of the Lindblad equation

Continuous measurement

- The interval $[0, T]$ is divided into N equipartitioned intervals
- The environmental orthonormal basis is $|m_{N-1}, \ldots, m_0\rangle$
- $|m_k\rangle$ interacts with S within the time interval $[t_k, t_{k+1}]$ via a unitary operator U_{t_k}



TUR for continuous measurement

■ One-step time evolution is

$$\rho(t+\Delta t) = \sum_c X_c \rho(t) X_c^\dagger$$

Kraus representation

where

$$X_0 \equiv e^{-i\Delta t H}\sqrt{\mathbb{I}_S - \Delta t \sum_c L_c^\dagger L_c} \quad \text{(no detection)}$$

$$X_c \equiv e^{-i\Delta t H}\sqrt{\Delta t} L_c \quad (\text{detection of } c^{\text{th}} \text{ event})$$

■ Because $V_0 \equiv \langle 0|U|0\rangle$ corresponds to "no jump events" during $[0,T]$, it is given by $V_0 = \lim_{N\to\infty} X_0^N$

TUR for continuous measurement

- V_0 can be computed via Trotter product formula as follows

$$V_0 = e^{-T\left(iH+\frac{1}{2}\sum_c L_c^\dagger L_c\right)}$$

Therefore, a quantum TUR becomes

$$\frac{\mathrm{Var}[g(m)]}{\langle g(m)\rangle^2} \geq \frac{1}{\Xi}$$

$$\Xi = \mathrm{Tr}_S\left[e^{T\left(iH+\frac{1}{2}\sum_c L_c^\dagger L_c\right)} e^{T\left(-iH+\frac{1}{2}\sum_c L_c^\dagger L_c\right)}\right] - 1$$

- This relation holds for any Lindblad dynamics (time-independent H and L_c) and for any initial density operator
- Ξ reduces to the dynamical activity in classical Markov processes in a particular limit

Effect of quantumness

- When we emulate classical Markov processes with the Lindblad equation, $\left[H, \sum_c L_c^\dagger L_c\right] = 0$ holds. In this case, Ξ reduces to

$$\Xi_{\mathrm{CL}} = \mathrm{Tr}_S\left[e^{T\sum_c L_c^\dagger L_c}\right]$$

- When $T \ll 1$, we have

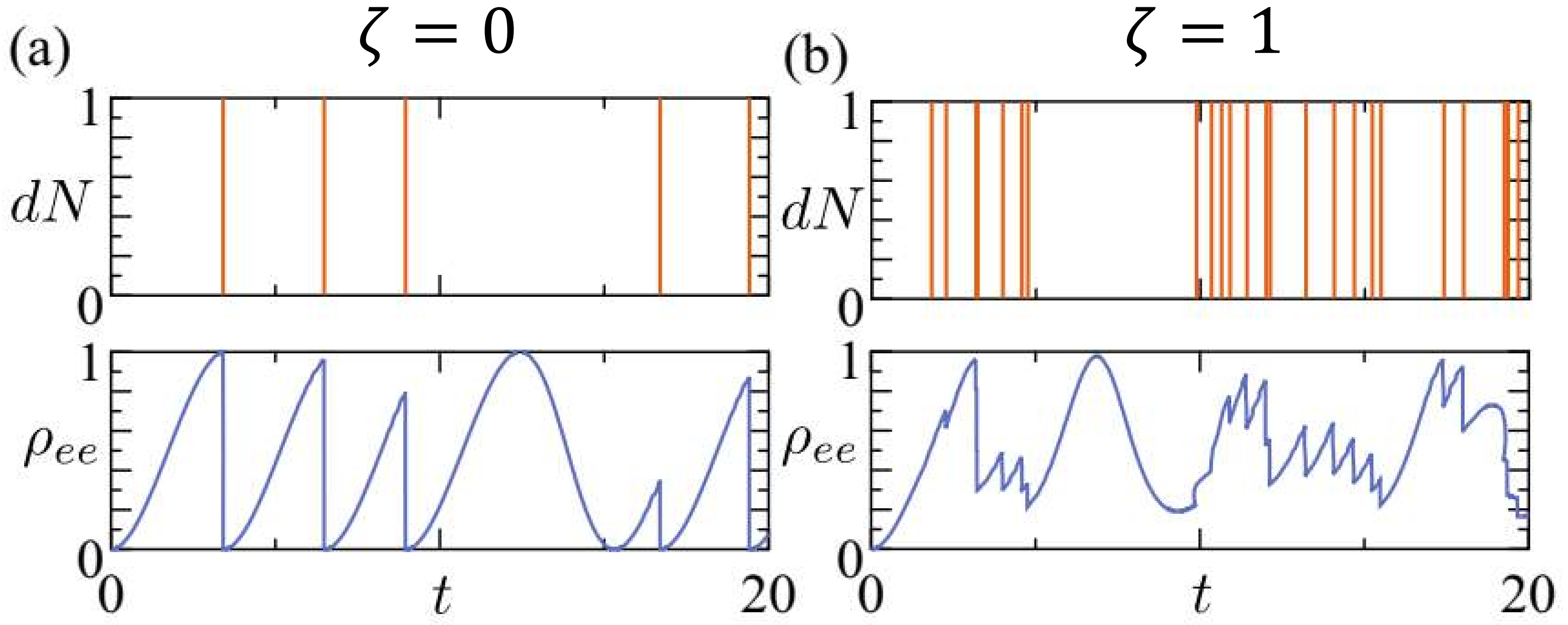
$$\Xi = \Xi_{\mathrm{CL}} + \frac{1}{2}T^2\chi + O(T^3)$$

where $\chi \equiv i\sum_c \mathrm{Tr}_S\left[[H, L_c^\dagger L_c]\rho\right]$.

- When $\chi > 0$, the system gains a precision enhancement due to the quantumness.
- For a particular model, χ corresponds to non-diagonal elements in density operators

Effect of measurement

- The Lindblad equation is invariant under the following transformation, $H \to H - \frac{i}{2}(\zeta^* L - \zeta L^\dagger)$, $L \to L + \zeta$ where ζ is an arbitrary complex parameter.
- Unravelling with different ζ corresponds to different continuous measurement



Both quantum trajectories reduce to the same dynamics on average

Effect of measurement

- Under this transformation, Ξ is

$$\Xi = \mathrm{e}^{|\zeta|^2 T}\mathrm{Tr}_S\left[e^{T\left(iH+\frac{1}{2}L^\dagger L+\zeta^* L\right)}e^{T\left(-iH+\frac{1}{2}L^\dagger L+\zeta L^\dagger\right)}\right]-1$$

- Therefore, for $|\zeta| \to \infty$, $\Xi \sim e^{|\zeta|^2 T}$
- The lower bound of the quantum TUR can be arbitrary small by employing a continuous measurement with large $|\zeta|$

- Measurements can be a thermodynamics resource. It is possible to extract work from single reservoir without feedback [Yi *et al*., PRE, 2017].

Classical limit and dynamical activity

- For classical Markov processes with transition rate $\gamma_{ji}(t)$ (from i to j) with the initial probability distribution P_i, Ξ becomes

$$\frac{\mathrm{Var}[g(m)]}{\langle g(m)\rangle^2} \geq \frac{1}{\Xi}, \qquad \Xi = \sum_i P_i \exp\left[\sum_{j\neq i}\int_0^T dt\, \gamma_{ji}(t)\right] - 1$$

- This relation holds for any time-dependent Markov chains
- For $T \ll 1$, we have

$$\Xi \simeq T \sum_i \sum_{j\neq i} P_i \gamma_{ji}(t)$$

which is the dynamical activity in classical Markov processes

- Dynamical activity plays important roles in classical Markov processes [Shiraishi *et al*., PRL, 2018], [Garrahan, PRE, 2017]

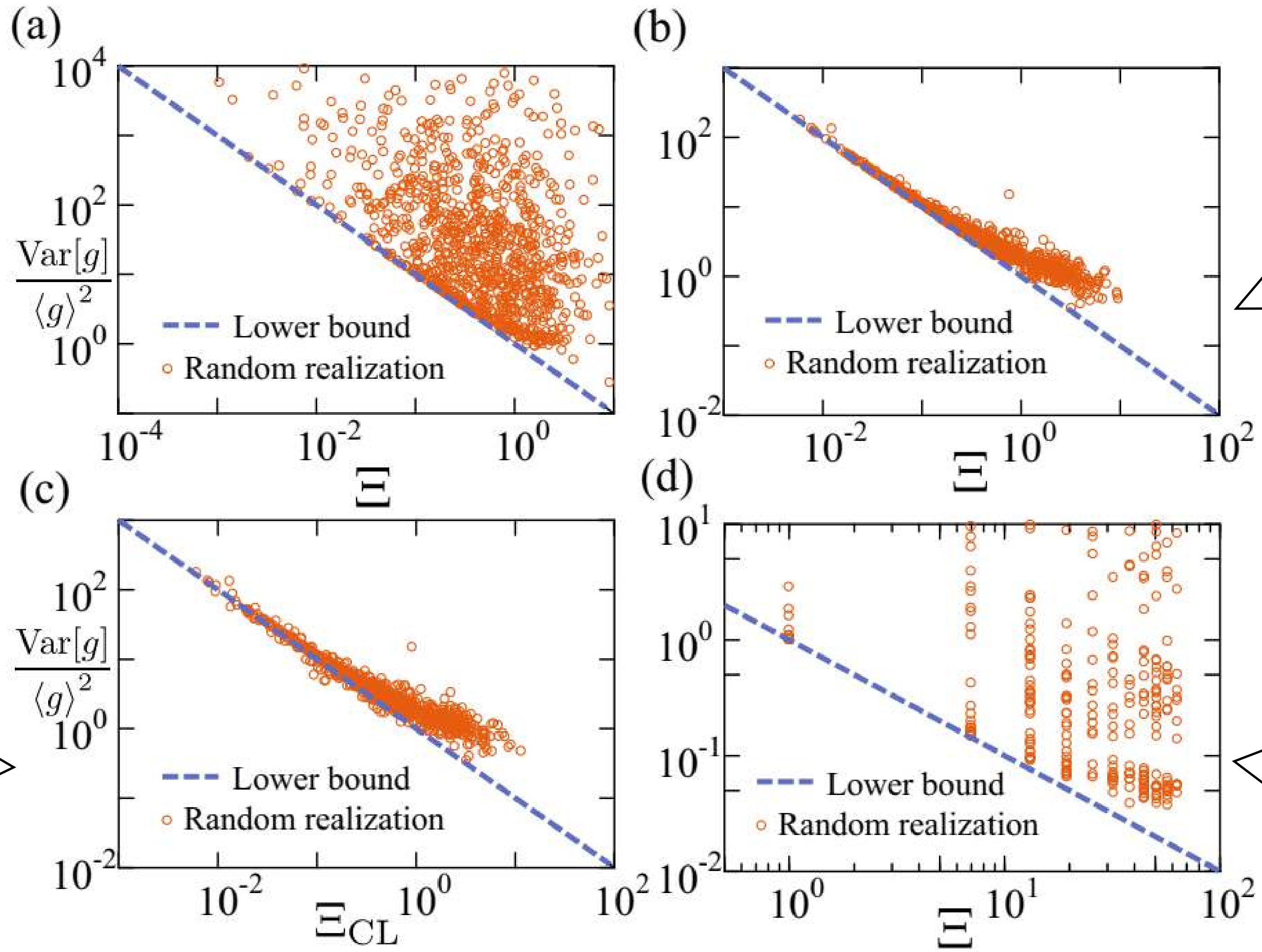
Numerical verifications

Random quantum channel and random $g(m)$

Continuous measurement in two-level atom driven by laser field. Observable is the number of emitted photon.

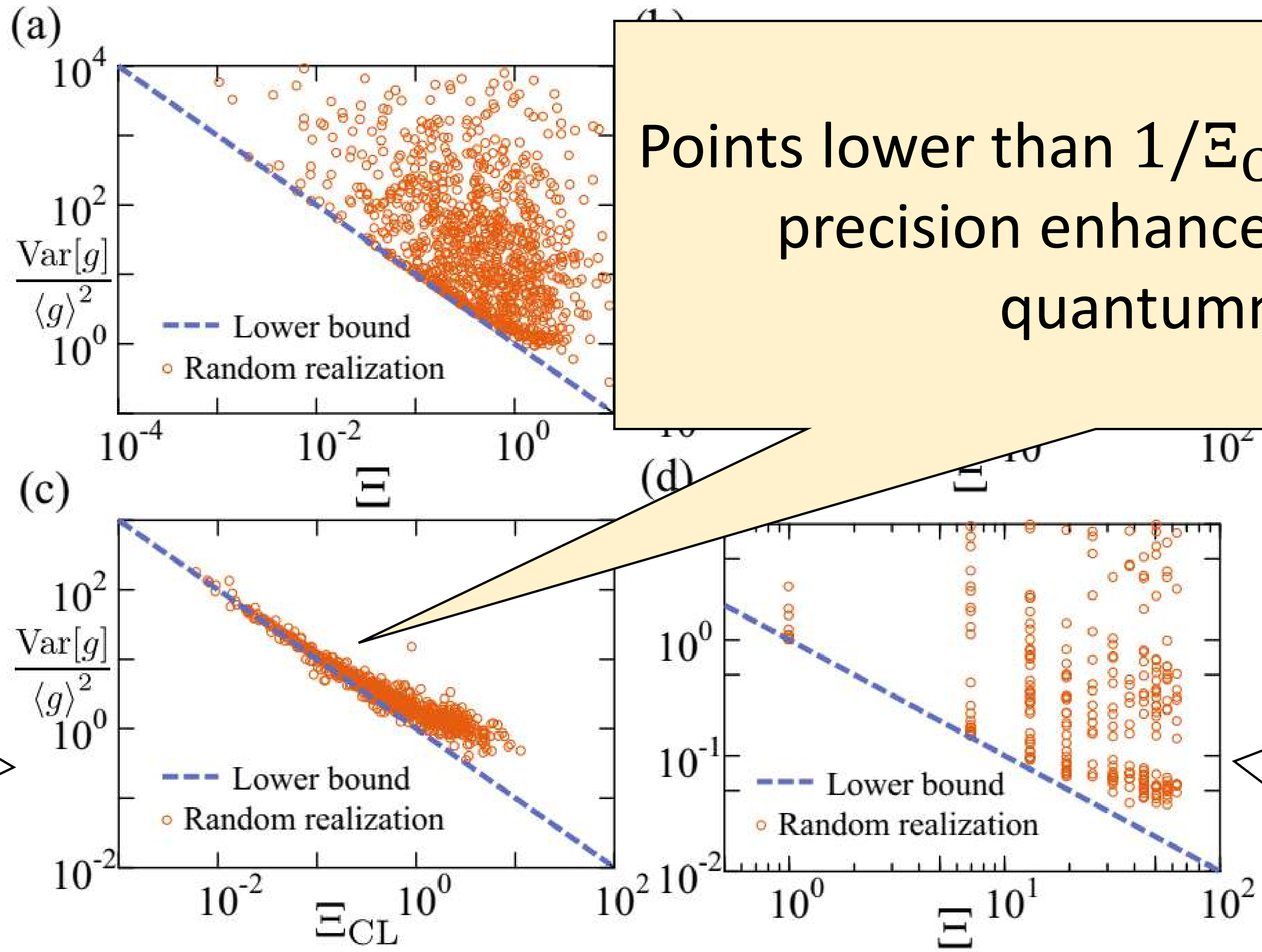
The same as (b). Ξ is replaced with Ξ_{CL}.

Quantum walk and random $g(m)$



Numerical verifications

Random quantum channel and random $g(m)$



Points lower than $1/\Xi_{\mathrm{CL}}$ is a signature of precision enhancement due to quantumness.

The same as (b). Ξ is replaced with Ξ_{CL}.

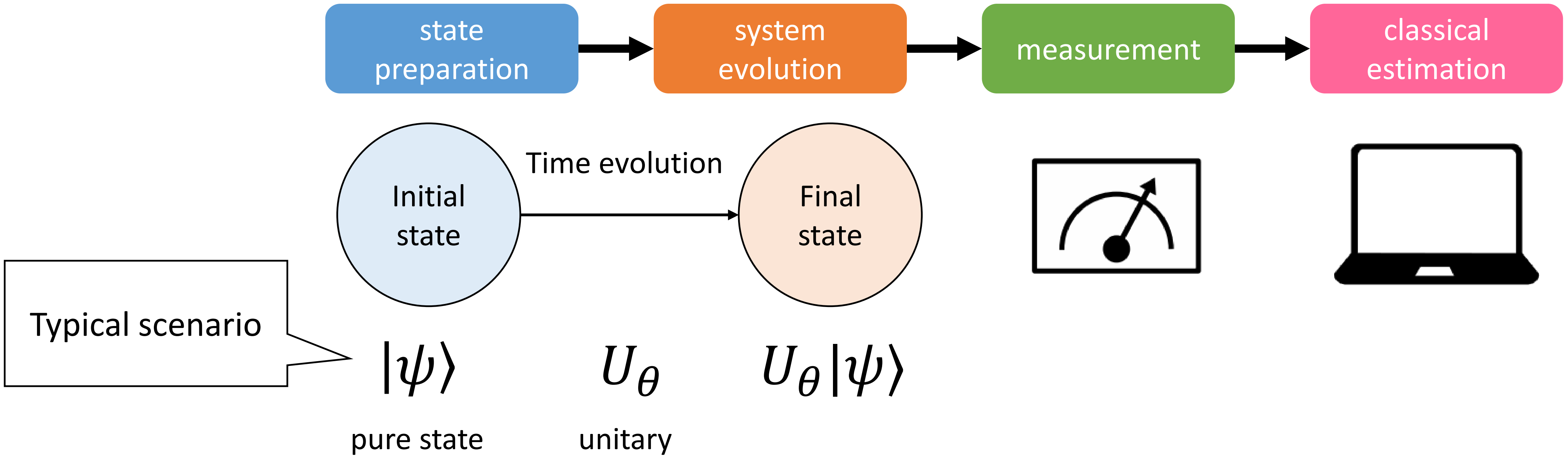
Quantum walk and random $g(m)$

Quantum Cramér-Rao inequality

■ Bound on statistical estimator in quantum systems

$$\mathrm{Var}[\hat{\theta}] \geq \frac{1}{\mathcal{F}_Q(\theta)}$$

$$\frac{\mathrm{Var}[\hat{\Theta}\,(\theta)]}{\left(\partial_\theta \langle \hat{\Theta} \rangle\right)^2} \geq \frac{1}{\mathcal{F}_Q(\theta)}$$



Quantum Cramér-Rao bound

- Quantum Cramér-Rao bound has been applied to obtain quantum uncertainty relations
  - Robertson uncertainty relation, Quantum speed limit
- Classical Cramér-Rao inequality has been applied to obtain classical thermodynamic uncertainty relations
  - [Hasegawa *et al*., PRE, 2019], [Dechant, JPA, 2019], [Ito *et al*. PRX, in press]
- It is much harder to find quantum Fisher information than in classical cases

Quantum Fisher information

- Quantum Fisher information is

$$\mathcal{F}_Q(\theta) = \max_{\mathcal{M}} \mathcal{F}_C(\theta; \mathcal{M})$$

where $\mathcal{M}$ is POVM and $\mathcal{F}_C$ is a classical Fisher information

- Therefore, quantum Cramér-Rao inequality is satisfied for any quantum measurements (POVMs)
- Quantum Fisher information is calculated by

$$\mathcal{F}_Q(\theta) = \mathrm{Tr}[\mathcal{L}^2 \rho]$$

where $\mathcal{L}$ is known as symmetric logarithm derivative.

- In general, $\mathcal{L}$ is difficult to obtain

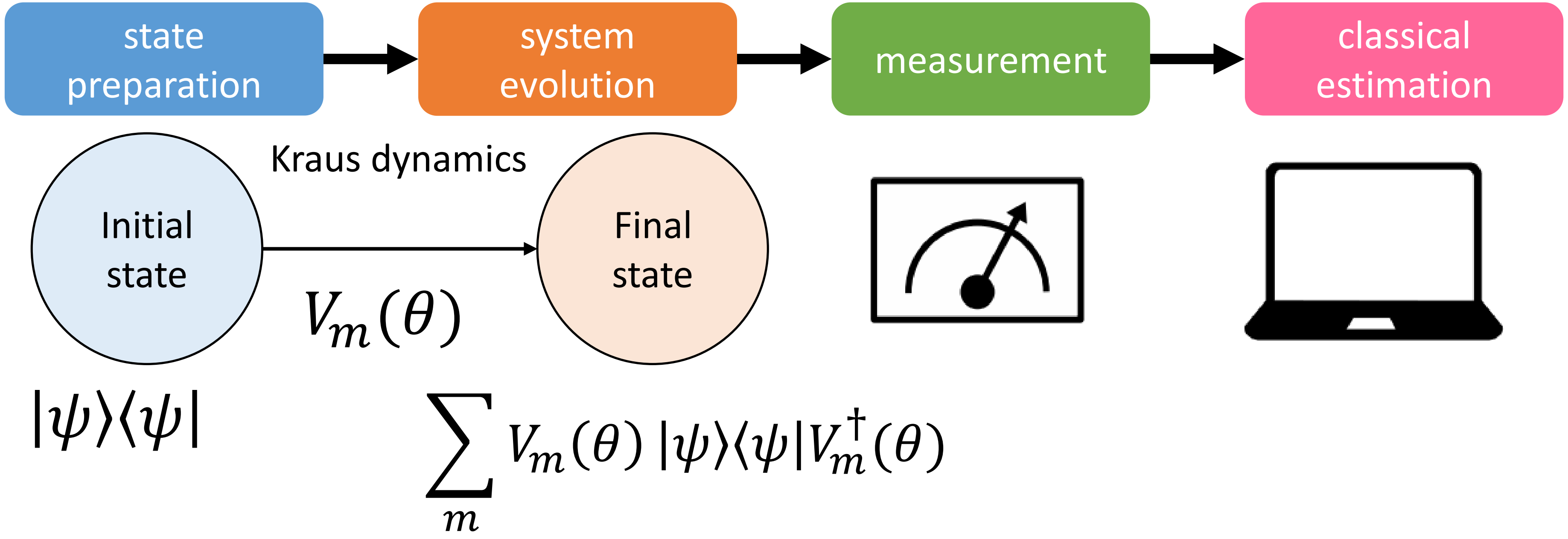
Quantum Fisher information

- [Escher *et al*, Nat. Phys., 2011] showed that quantum Fisher information is upper bounded by

$$\mathcal{F}_Q(\theta) \le 4[\langle\psi|H_1(\theta)|\psi\rangle - \langle\psi|H_2(\theta)|\psi\rangle^2]$$

where

$$H_1(\theta) = \sum_{m=0}^{M-1} \frac{\partial V_m^\dagger(\theta)}{\partial\theta}\frac{\partial V_m(\theta)}{\partial\theta}, H_2(\theta) = i\sum_{m=0}^{M-1}\frac{\partial V_m^\dagger(\theta)}{\partial\theta}V_m(\theta)$$



Derivation

- To derive the main result, we consider the following parametrization

$$V_m(\theta) = e^{\theta/2} V_m \quad (1 \leq m \leq M-1)$$

- We cannot freely parametrize $V_0(\theta)$ due to the completeness relation

$$\sum_{m=0}^{M-1} V_m^\dagger(\theta) V_m(\theta) = \mathbb{I}$$

- Any $V_0(\theta)$ satisfying the completeness relation can be represented by

$$V_0(\theta) = Y \sqrt{\mathbb{I} - \sum_{m=1}^{M-1} V_m^\dagger(\theta) V_m(\theta)} = Y \sqrt{\mathbb{I} - e^\theta \sum_{m=1}^{M-1} V_m^\dagger V_m}$$

where Y is a unitary operator.

Derivation

- Using these parametrization, QFI is upper bounded by

$$\mathcal{F}_Q(\theta=0) \le \left\langle\psi\middle|\left(V_0^\dagger V_0\right)^{-1}\middle|\psi\right\rangle - 1$$

- We next evaluate $\partial_\theta \langle g(m)\rangle_\theta$ in quantum Cramér-Rao inequality
- Since we have assumed that $g(0)=0$, a complicated scaling dependence of $V_0(\theta)$ on θ can be ignored

$$\langle g(m)\rangle_\theta = \sum_{m=0}^{M-1} \langle\psi|V_m^\dagger(\theta)V_m(\theta)|\psi\rangle g(m)$$

$$= \sum_{m=1}^{M-1} \langle\psi|V_m^\dagger(\theta)V_m(\theta)|\psi\rangle g(m)$$

$$= e^\theta \langle g(m)\rangle_{\theta=0}$$

Derivation

- Substituting into these equality to the quantum Cramér-Rao inequality, we obtain the main result

$$\frac{\mathrm{Var}[g(m)]}{\langle g(m)\rangle^2} \geq \frac{1}{\left\langle \psi \middle| \left(V_0^\dagger V_0\right)^{-1} \middle| \psi \right\rangle - 1}$$

- The main result also holds for any initial mixed states ρ through the purification

$$\frac{\mathrm{Var}[g(m)]}{\langle g(m)\rangle^2} \geq \frac{1}{\mathrm{Tr}\left[\left(V_0^\dagger V_0\right)^{-1} \rho\right] - 1}$$

Conclusion

- TUR in open quantum systems is obtained
  - Quantum dynamics = Joint unitary evolution on principal and environment systems
  - Observable = Projective measurement on the environment
  - Thermodynamic cost = Quantum analogue of dynamical activity

- Effects of quantumness on precision
  - Measurements improve the precision
  - Non-commutativeness improves the precision

Questions?

Quantum walk

- The quantum walk is defined on the chirality space spanned by $\{|R\rangle, |L\rangle\}$ and the position space spanned by $\{|n\rangle\}$
- One step evolution is operated via a unitary operator

$$\mathcal{U} = \mathcal{S}(\mathcal{C} \otimes \mathbb{I}_E)$$

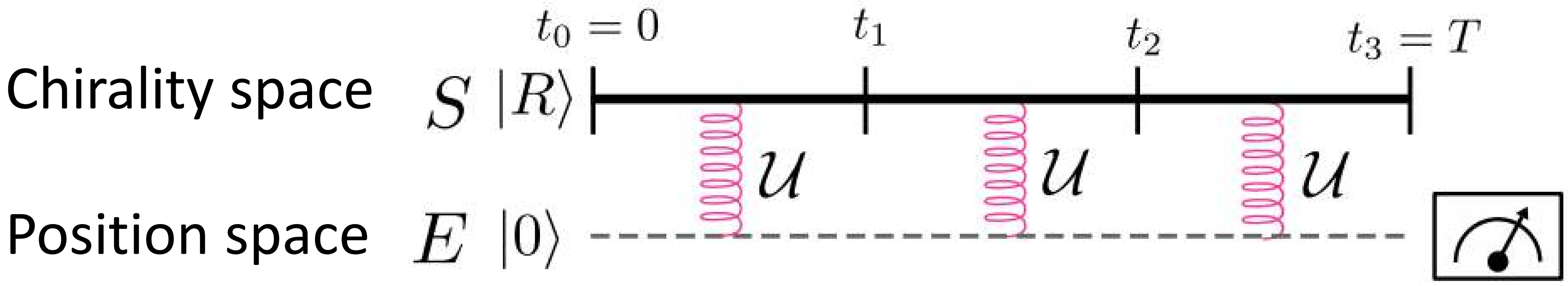
where

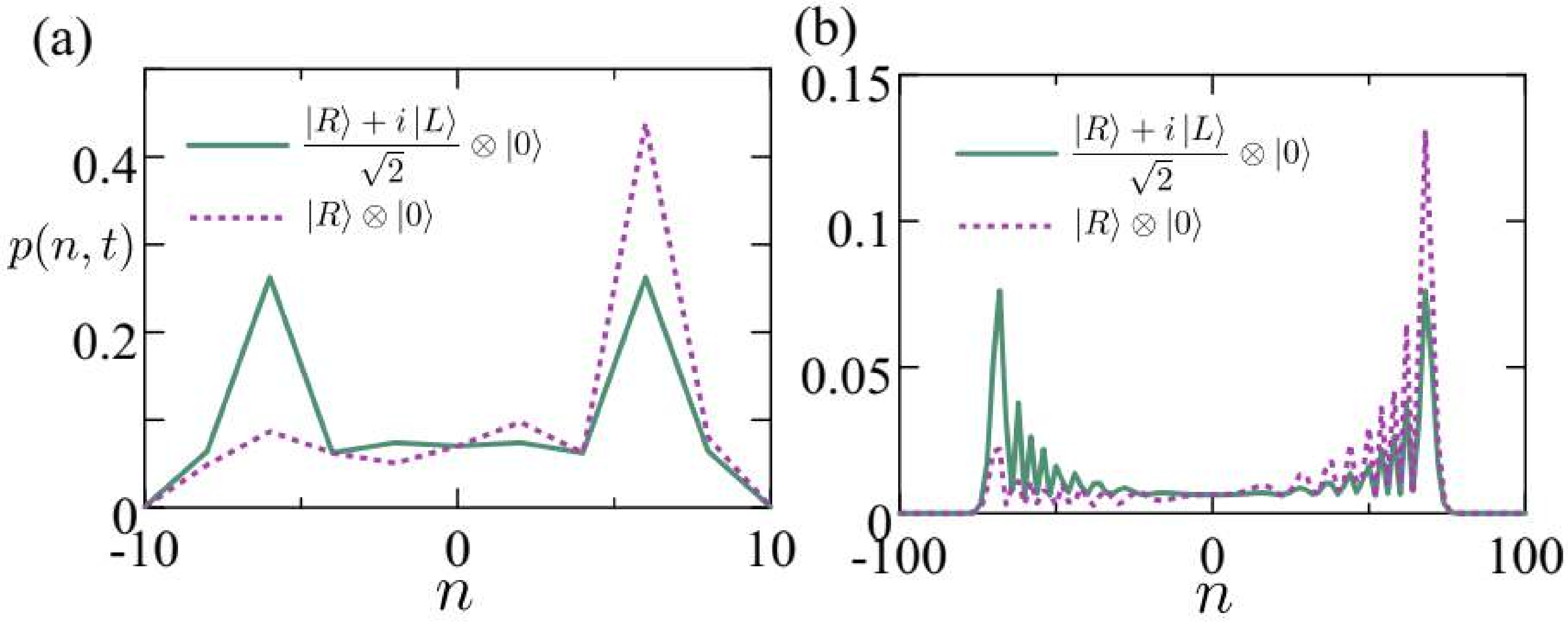
$$\mathcal{C} \equiv \frac{|R\rangle\langle R| + |R\rangle\langle L| + |L\rangle\langle R| - |L\rangle\langle L|}{\sqrt{2}}$$

Hadamard gate

$$\mathcal{S} = \sum_n [|R\rangle\langle R| \otimes |n+1\rangle\langle n| + |L\rangle\langle L| \otimes |n-1\rangle\langle n|]$$

Quantum walk





Quantum walk

- The amplitudes after t steps were known
- Therefore, dynamical activity Ξ after t steps can be calculated analytically

$$\Xi = \begin{cases} 2^{2u+1}\binom{u}{\frac{u}{2}}^{-2} - 1 & u \in \text{even} \\ 2^{2u-1}\binom{u-1}{\frac{u-1}{2}}^{-2} - 1 & u \in \text{odd} \end{cases}$$

where $u \equiv \frac{t}{2}$

- By using Stirling approximation,

$$\Xi \sim \pi u$$

Quantum walk

- Ξ linearly depends on the number of steps.
- This is in contrast to the classical case where Ξ exponentially depends on time