The Error–Dissipation Tradeoff when Computing with Time-Symmetric Protocols

Paul M. Riechers

in collaboration with
Alec B. Boyd, Gregory W. Wimmsat,
James P. Crutchfield, and Mile Gu

Complexity Institute, and School of Physical and Mathematical Sciences
Nanyang Technological University, Singapore

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What are the thermodynamic limits of computation?
What are the thermodynamic limits of computation?

**Landauer’s bound?**

- $k_B T \ln 2$ of heat to compress state space in a binary reset

Essentially restates the Second Law of thermodynamics—that entropy production is expected to be non-negative: $\langle \Sigma \rangle \geq 0$

$\implies$ The bound can only be achieved in the quasistatic limit, and assumes no control restrictions
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What are the *nonequilibrium* thermodynamic limits of *realistic* computation?
What are the *nonequilibrium* thermodynamic limits of *realistic* computation?

*Nonequilibrium*: Finite duration

*Realistic*: Control constraints

**Landauer’s bound?**

Stronger constraints on entropy production?
What are the *nonequilibrium* thermodynamic limits of *realistic* computation?

**Nonequilibrium**: Finite duration

**Realistic**: Control constraints

We will show that metastable memories and *time-symmetric control* together imply an *error–dissipation tradeoff* for any computation $C$. 
What are the *nonequilibrium* thermodynamic limits of *realistic* computation?

*Nonequilibrium*: Finite duration

*Realistic*: Control constraints

We will show that *metastable memories* and *time-symmetric control* together imply an *

error–dissipation tradeoff* for any computation \( C \)

\[
\langle \Sigma^{t\text{-sym}}_{\text{min}} \rangle \sim f(C, \dagger) \ln(1/\epsilon)
\]

where \( \epsilon \) is the error tolerance, and \( \dagger \) describes the time-reversal symmetries of the memory elements
non-volatile memory

\[ \dot{Q}_{hk} = 0 \]

volatile memory

\[ \dot{Q}_{hk} > 0 \]

Consider non-volatile memory.

(Add housekeeping penalty for volatile memory.)
Memory must be physically embedded. Memories partition the microstates of a physical memory system.

\[ m \subset S, \quad \bigcup_{m \in M} m = S \]

Memory is stored metastably between computations.
Computation is the transformation of memory.

Physical transformation via ephemeral control $x_0:τ$

Physical transformation implements stochastic computation $p(M_τ|M_0)$

Physical transformation implements stochastic computation \( p(\mathcal{M}_\tau | \mathcal{M}_0) \)

In nearly-deterministic case:
\( \mathcal{C} : \mathcal{M} \rightarrow \mathcal{M} \) with error tolerance \( \epsilon \)

Metastable Memory Storage

Memories are metastable between computations if

\[ \mu_0^{[m]} \approx \mu_{\tau}^{[m]} \approx \pi_{x_0}^{[m]} \]

Then, \( \mu_t \approx \sum_{m \in \mathcal{M}} \mu_t(m) \pi_{x_0}^{[m]} \) at \( t = 0, \tau \)

Computation is the transformation of memory

Example: Erasure to $L$

\[ \mu_t \approx \sum_{m \in \mathcal{M}} \mu_t(m) \pi_{x_0}^m \] at $t = 0, \tau$
Main results can be derived in either classical or quantum setting (will stick with classical limit here)

- System is prepared with an initial distribution $\mu_0$ over microstates
- Each bath is initially in local equilibrium
- Nonequilibrium driving $x_{0:\tau}$ controls Hamiltonian of joint system–baths megasystem
General Thermodynamic Setup

**Entropy Flow:** \( \Phi_{\text{env}} = \sum_{b \in \mathcal{B}} \frac{Q^{(b)}}{T^{(b)}} - \frac{1}{T^{(b)}} \sum_{k} \mu^{(b)}_{k} \Delta N^{(b)}_{k}, \)

**Surprisal of System State:** \( S_{\text{sys}} = -k_{B} \ln \mu_{t}(s_{t}) \)

**Entropy Production:**
\[
\Sigma = \Phi_{\text{env}} + \Delta S_{\text{sys}}
\]

**2nd Law**
\( \langle \Sigma \rangle \geq 0 \)

**Landauer’s bound**
\( \langle \Phi_{\text{env}} \rangle \geq -\Delta \langle S_{\text{sys}} \rangle \)
Tighter than Landauer’s bound

Thermodynamic Implications of Memory Transitions

the Detailed Fluctuation Theorem

\[
\frac{\Pr_{x_0:\tau}(S_{0:\tau} = s_{0:\tau}, \Phi_{\text{env}} = \phi | S_0 = s_0)}{\Pr_{\mathcal{X}(x_0:\tau)}(\tilde{S}_{0:\tau} = \mathcal{X}(s_{0:\tau}), \tilde{\Phi}_{\text{env}} = -\phi | \tilde{S}_0 = s^\dagger_\tau)} = e^{\phi/k_B}
\]

(Jarzynski, JStatPhys, 2000)

\[
s_{0:\tau} = s_0 s_\tau / N \cdots s_\tau
\]

non-Markovianity is non-problematic
Tighter than Landauer’s bound

Thermodynamic Implications of Memory Transitions

The Detailed Fluctuation Theorem

\[
\frac{\Pr_{x_0:τ}(S_{0:τ} = s_{0:τ}, Φ_{env} = φ|S_0 = s_0)}{\Pr_{Y(x_0:τ)}(\tilde{S}_{0:τ} = Y(s_{0:τ}), \tilde{Φ}_{env} = -φ|\tilde{S}_0 = s_↑)} = e^{φ/k_B}
\]

(Jarzynski, JStatPhys, 2000)

E.g.,
\[
s = (\vec{q}, \vec{φ}) \rightarrow s↑ = (\vec{q}, -\vec{φ})
\]
Thermodynamic Implications of Memory Transitions

the Detailed Fluctuation Theorem

\[
\frac{\Pr_{x_0:\tau}(S_{0:\tau} = s_{0:\tau}, \Phi_{\text{env}} = \phi|S_0 = s_0)}{\Pr_{\mathcal{R}(x_0:\tau)}(\tilde{S}_{0:\tau} = \mathcal{R}(s_{0:\tau}), \tilde{\Phi}_{\text{env}} = -\phi|\tilde{S}_0 = s^\dagger_\tau)} = e^{\phi/k_B}
\]

(Jarzynski, JStatPhys, 2000)

\[
\to \Sigma = k_B \ln \frac{\mu_0(s_0) \Pr_{x_0:\tau}(S_{0:\tau} = s_{0:\tau}, \Phi_{\text{env}} = \phi|S_0 = s_0)}{\mu_\tau(s_\tau) \Pr_{\mathcal{R}(x_0:\tau)}(\tilde{S}_{0:\tau} = \mathcal{R}(s_{0:\tau}), \tilde{\Phi}_{\text{env}} = -\phi|\tilde{S}_0 = s^\dagger_\tau)}
\]
Thermodynamic Implications of Memory Transitions

the Detailed Fluctuation Theorem

\[
\frac{\Pr_{x_0:\tau}(S_{0:\tau} = s_{0:\tau}, \Phi_{\text{env}} = \phi|S_0 = s_0)}{\Pr_{\mathcal{R}(x_0:\tau)}(\tilde{S}_{0:\tau} = \mathcal{R}(s_{0:\tau}), \tilde{\Phi}_{\text{env}} = -\phi|\tilde{S}_0 = s^\dagger_\tau)} = e^{\phi/k_B}
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(Jarzynski, JStatPhys, 2000)

\[
\rightarrow \Sigma = k_B \ln \frac{\mu_0(s_0) \Pr_{x_0:\tau}(S_{0:\tau} = s_{0:\tau}, \Phi_{\text{env}} = \phi|S_0 = s_0)}{\mu_{\tau}(s_\tau) \Pr_{\mathcal{R}(x_0:\tau)}(\tilde{S}_{0:\tau} = \mathcal{R}(s_{0:\tau}), \tilde{\Phi}_{\text{env}} = -\phi|\tilde{S}_0 = s^\dagger_\tau)}
\]

\[
= k_B \ln \frac{\rho(S_{0:\tau} = s_{0:\tau})}{\rho^R(S_{0:\tau} = s_{0:\tau})} + k_B \ln \frac{\mathcal{P}(\Phi_{\text{env}} = \phi|S_{0:\tau} = s_{0:\tau})}{\mathcal{R}(\Phi_{\text{env}} = \phi|S_{0:\tau} = s_{0:\tau})}
\]
a new and useful decomposition for entropy production

\[ \Sigma = k_B \ln \frac{\rho(S_{0:\tau} = s_{0:\tau})}{\rho_R(S_{0:\tau} = s_{0:\tau})} + k_B \ln \frac{\mathcal{P}(\Phi_{env} = \phi|S_{0:\tau} = s_{0:\tau})}{\mathcal{R}(\Phi_{env} = \phi|S_{0:\tau} = s_{0:\tau})} \]

where

\[ \rho(S_{0:\tau} = s_{0:\tau}) \equiv \Pr_{x_{0:\tau}}(S_{0:\tau} = s_{0:\tau}|S_0 \sim \mu_0) \]

\[ \rho_R(S_{0:\tau} = s_{0:\tau}) \equiv \Pr_{\mathcal{R}(x_{0:\tau})}(\tilde{S}_{0:\tau} = \mathcal{R}(s_{0:\tau})|\tilde{S}_0 \sim \mu^\dagger_\tau) \]

\[ \mu^\dagger_\tau(s) = \mu_\tau(s^\dagger) \]

\[ \mathcal{P}(\Phi_{env} = \phi|S_{0:\tau} = s_{0:\tau}) \equiv \Pr_{x_{0:\tau}}(\Phi_{env} = \phi|S_{0:\tau} = s_{0:\tau}) \]

\[ \mathcal{R}(\Phi_{env} = \phi|S_{0:\tau} = s_{0:\tau}) \equiv \Pr_{\mathcal{R}(x_{0:\tau})}(\tilde{\Phi}_{env} = -\phi|S_{0:\tau} = \mathcal{R}(s_{0:\tau})) \]
a new and useful decomposition for entropy production

\[ \Sigma = k_B \ln \frac{\rho(S_{0:\tau} = s_{0:\tau})}{\rho^R(S_{0:\tau} = s_{0:\tau})} + k_B \ln \frac{P(\Phi_{env} = \phi | S_{0:\tau} = s_{0:\tau})}{R(\Phi_{env} = \phi | S_{0:\tau} = s_{0:\tau})} \]

\[ \frac{1}{k_B} \langle \Sigma \rangle = D_{KL}[\rho(S_{0:\tau}) \parallel \rho^R(S_{0:\tau})] + D_{KL}[P(\Phi_{env} | S_{0:\tau}) \parallel R(\Phi_{env} | S_{0:\tau})] \]
a new and useful decomposition for entropy production

$$\Sigma = k_B \ln \frac{\rho(S_{0:\tau} = s_{0:\tau})}{\rho^R(S_{0:\tau} = s_{0:\tau})} + k_B \ln \frac{\mathcal{P}(\Phi_{env} = \phi|S_{0:\tau} = s_{0:\tau})}{\mathcal{R}(\Phi_{env} = \phi|S_{0:\tau} = s_{0:\tau})}$$

$$\frac{1}{k_B} \langle \Sigma \rangle = D_{KL} \left[ \rho(S_{0:\tau}) \left| \rho^R(S_{0:\tau}) \right. \right] + D_{KL} \left[ \mathcal{P}(\Phi_{env}|S_{0:\tau}) \left| \mathcal{R}(\Phi_{env}|S_{0:\tau}) \right. \right]$$

$$\geq D_{KL} \left[ \rho(S_{0:\tau}) \left| \rho^R(S_{0:\tau}) \right. \right]$$
This is importantly distinct from superficially similar previous results from Jarzynski ‘06, Kawai ‘07, Gomez-Marin ‘08, Parrondo ‘09, Roldan ‘10, etc. (Previous results assumed either initial equilibrium—which wipes out memory—or NESS.)

Dependence on $\mu_0$ and $\mu_\tau^\dagger$ is crucial to address thermodynamics of general memory transformations.
Coarse grain over both time and state space.

Microstate trajectories

\[ \Sigma = k_B \ln \frac{\rho(S_{0:}\tau = s_{0:}\tau)}{\rho^R(S_{0:}\tau = s_{0:}\tau)} + k_B \ln \frac{\mathcal{P}(\Phi_{\text{env}} = \phi|S_{0:}\tau = s_{0:}\tau)}{\mathcal{R}(\Phi_{\text{env}} = \phi|S_{0:}\tau = s_{0:}\tau)} \]

\[
\frac{1}{k_B} \langle \Sigma \rangle = D_{\text{KL}} \left[ \rho(S_{0:}\tau) \left\| \rho^R(S_{0:}\tau) \right\] + D_{\text{KL}} \left[ \mathcal{P}(\Phi_{\text{env}} | S_{0:}\tau) \left\| \mathcal{R}(\Phi_{\text{env}} | S_{0:}\tau) \right\] \\
\geq D_{\text{KL}} \left[ \rho(S_{0:}\tau) \left\| \rho^R(S_{0:}\tau) \right\] \\
\geq D_{\text{KL}} \left[ \rho(M_0, M_\tau) \left\| \rho^R(M_0, M_\tau) \right\] .
\]
Tighter than Landauer’s bound

**Requirement for dissipationless computing**

Physics, meet computation:

\[
\frac{1}{k_B} \langle \Sigma \rangle \geq D_{KL} \left[ \underbrace{\rho(M_0, M_\tau)}_{\mu_0(m) \ Pr(\mu_0^m \ x_0:\tau \rightarrow m')} \left| \underbrace{\rho^R(M_0, M_\tau)}_{\mu_\tau(m') \ Pr(\mu_\tau^{m'} \ x_0:\tau \rightarrow m^\dagger)} \right] \right]
\]

Zero dissipation requires *density reversibility*:

\[
\Pr(\mu_\tau^{m'} \ x_0:\tau \rightarrow m^\dagger) = \frac{\mu_0(m) \ p(m \rightarrow m')}{\mu_\tau(m')}.
\]

Tighter than Landauer’s bound

**Requirement for dissipationless computing**

Physics, meet computation:

\[
\frac{1}{k_B} \langle \Sigma \rangle \geq D_{\text{KL}} \left[ \rho(\mathcal{M}_0, \mathcal{M}_\tau) \parallel \rho^R(\mathcal{M}_0, \mathcal{M}_\tau) \right] \\
\mu_0(m) \Pr(\mu_0^{[m]} \xrightarrow{x_0: \tau} m') \quad \mu_\tau(m') \Pr(\mu_\tau^{[m' \dagger]} \xrightarrow{\mathcal{A}(x_0: \tau)} m^\dagger)
\]

Zero dissipation requires *density reversibility*:

\[
\Pr(\mu_\tau^{[m' \dagger]} \xrightarrow{\mathcal{A}(x_0: \tau)} m^\dagger) = \frac{\mu_0(m) \ p(m \rightarrow m')}{\mu_\tau(m')}. \\
\text{physics} \quad \text{computation}
\]

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<tr>
<th>Positional Storage</th>
<th>Spin Storage</th>
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| (\text{\textup{\textup{\texttt{W}}}})^\dagger = \text{\textup{\textup{\texttt{W}}}} | (\text{\textup{\textup{\texttt{\textcircled{L}}}}})^\dagger = \text{\textup{\textup{\texttt{\textcircled{L}}}}}|}
| (\text{\textup{\textup{\texttt{\textcircled{W}}}}})^\dagger = \text{\textup{\textup{\texttt{\textcircled{W}}}}}| (\text{\textup{\textup{\texttt{\textcircled{\textcircled{L}}}}}})^\dagger = \text{\textup{\textup{\texttt{\textcircled{\textcircled{L}}}}}}|

Time-reversal symmetry of memory elements matters for thermodynamics!
Often, practical constraints require time-symmetric control.
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E.g., Biological transformations at very low Reynolds numbers!
Often, practical constraints require time-symmetric control.

E.g., Biological transformations at very low Reynolds numbers!

This time-symmetric signal drives all logical transformations in a computer (program, input memory, working memory, etc. are all part of memory system)
Entropy production required for time-symmetric control of metastable memories:

\[
\frac{1}{k_B} \langle \Sigma \rangle \geq D_{KL} \left[ \mu_0(m) \Pr(\mu_0^m \xrightarrow{x_0: \tau} m') \right\| \mu_\tau(m') \Pr(\mu_\tau^{m'} \xrightarrow{\mathcal{A}(x_0: \tau)} m^\dagger) \right]
\]

(with probability elements representing their distributions)
Time-symmetric control of metastable memories

Entropy production required for
time-symmetric control of metastable memories:

\[
\mathcal{R}(x_{0;\tau}) = x_{0;\tau}
\]

\[
\frac{1}{k_B} \langle \Sigma \rangle \geq D_{\text{KL}} \left[ \mu_0(m) \Pr(\mu_0^{[m]} \xrightarrow{x_{0;\tau}} m') \left\| \mu_\tau(m') \Pr(\mu_\tau^{[m']} \xrightarrow{\mathcal{R}(x_{0;\tau})} m^\dagger) \right. \right] \]

Time-symmetric control of metastable memories

Entropy production required for time-symmetric control of metastable memories:

\[ \mathcal{H}(x_{0:\tau}) = x_{0:\tau} \]

\[ \frac{1}{k_B} \langle \Sigma \rangle \geq D_{\text{KL}} \left[ \mu_0(m) \Pr (\mu_0^{[m]} \xrightarrow{x_{0:\tau}} m') \left\| \mu_{\tau}(m') \Pr (\mu_{\tau}^{[m']} \xrightarrow{x_{0:\tau}} m^\dagger) \right. \right] \]
Time-symmetric control of metastable memories

Entropy production required for time-symmetric control of metastable memories:

\[ \mathcal{K}(x_{0:}\tau) = x_{0:\tau} \]

\[ \mu_0^*[m] = \pi_{x_0}, \quad \mu_\tau^*[m] = \pi_{x_\tau} \]

\[ \frac{1}{k_B} \langle \Sigma \rangle \geq D_{KL} \left[ \mu_0(m) \Pr(\mu_0^*[m] \xrightarrow{x_0:\tau} m') \bigg\| \mu_\tau(m') \Pr(\mu_\tau^*[m'^\dagger] \xrightarrow{x_0:\tau} m'^\dagger) \right] \]
Time-symmetric control

Time-symmetric control of metastable memories

Entropy production required for time-symmetric control of metastable memories:

\[ \mathcal{R}(x_{0:\tau}) = x_{0:\tau} \]

\[ \mu_0[m] = \pi_{x_0}, \quad \mu_{\tau}[m] = \pi_{x_\tau} \]

\[ \frac{1}{k_B} \langle \Sigma \rangle \geq D_{KL} \left[ \mu_0(m) \Pr\left( \pi_{x_0} \xrightarrow{x_{0:\tau}} m' \right) \bigg\| \mu_{\tau}(m') \Pr\left( \mu_{\tau}^{[m'\dagger]} \xrightarrow{x_{0:\tau}} m^{\dagger} \right) \right] \]
Time-symmetric control

Time-symmetric control of metastable memories

Entropy production required for time-symmetric control of metastable memories:

\[ \mathcal{R}(x_{0;\tau}) = x_{0;\tau} \]
\[ \mu_0 = \pi_{x_0}, \quad \mu_\tau = \pi_{x_\tau} \]

\[
\frac{1}{k_B} \langle \Sigma \rangle \geq D_{KL} \left[ \mu_0(m) \Pr(\pi_{x_0} \xrightarrow{x_{0;\tau}} m') \bigg\| \mu_\tau(m') \Pr(\mu_\tau[m'] \xrightarrow{x_{0;\tau}} m^\dagger) \right]
\]
Entropy production required for time-symmetric control of metastable memories:

\[ \mathcal{R}(x_{0:}\tau) = x_{0:}\tau \]

\[ \mu_0^{[m]} = \pi_{x_0}^{[m]}, \quad \mu_{\tau}^{[m]} = \pi_{x_\tau}^{[m]} \]

\[ \frac{1}{k_B} \langle \Sigma \rangle \geq D_{KL} \left[ \mu_0(m) \Pr(\pi_{x_0}^{[m]} \xrightarrow{x_{0:}\tau} m') \middle|\middle| \mu_{\tau}(m') \Pr(\pi_{x_\tau}^{[m']} \xrightarrow{x_{0:}\tau} m^+) \right] \]
Time-symmetric control

Time-symmetric control of metastable memories

Entropy production required for time-symmetric control of metastable memories:

\[
\mathcal{R}(x_{0: \tau}) = x_{0: \tau}
\]

\[
\mu_0[m] = \pi_{x_0}, \quad \mu[\tau][m] = \pi_{x_\tau}
\]

\[
\frac{1}{k_B} \langle \Sigma \rangle \geq D_{KL} \left[ \mu_0(m) \Pr(\pi_{x_0}[m] \xrightarrow{x_{0: \tau}} m') \left\| \mu[\tau][m'] \Pr(\pi_{x_\tau}[m^\dagger] \xrightarrow{x_{0: \tau}} m^\dagger) \right. \right]
\]
Entropy production required for time-symmetric control of metastable memories:

\[
\mathcal{Y}(x_{0:\tau}) = x_{0:\tau} \\
\mu_0[m] = \pi_{x_0}, \quad \mu_{\tau}[m] = \pi_{x_{\tau}}
\]

\[
\frac{1}{k_B} \langle \Sigma \rangle \geq D_{KL} \left[ \mu_0(m) \Pr(\pi_{x_0}^{[m]} \xrightarrow{x_0:\tau} m') \middle\| \mu_{\tau}(m') \Pr(\pi_{x_0}^{[m']^\dagger}] \xrightarrow{x_0:\tau} m^\dagger) \right]
\]
Time-symmetric control

Time-symmetric control of metastable memories

Entropy production required for time-symmetric control of metastable memories:

\[ \mathcal{R}(x_{0:\tau}) = x_{0:\tau} \]

\[ \mu_0 = \pi_{x_0}, \quad \mu_\tau = \pi_{x_\tau} \]

\[
\frac{1}{k_B} \langle \Sigma \rangle \geq D_{KL} \left[ \mu_0(m) \Pr(\pi_{x_0}^{[m]} \xrightarrow{x_{0:\tau}} m') \bigg\| \mu_\tau(m') \Pr(\pi_{x_0}^{[m']} \xrightarrow{x_{0:\tau}} m^\dagger) \right]
\]
Time-symmetric control of metastable memories

Entropy production required for time-symmetric control of metastable memories:

\[ \mathcal{R}(x_{0:\tau}) = x_{0:\tau} \]

\[ \mu_0^{[m]} = \pi_{x_0}^{[m]}, \quad \mu_\tau^{[m]} = \pi_{x_\tau}^{[m]} \]

\[ \frac{1}{k_B} \langle \Sigma \rangle \geq D_{KL} \left[ \mu_0^{[m]} p(m \to m') \left\| \mu_\tau^{[m']} p(m'^{\dagger} \to m^{\dagger}) \right\| \right] \]

where \( \{p(m \to m')\}_{m,m'} \) are the actual transition probabilities between memory states of the computer!
Time-symmetric control of metastable memories

Entropy production required for time-symmetric control of metastable memories:

\[ \mathcal{R}(x_{0:\tau}) = x_{0:\tau} \quad \mu_0^{[m]} = \pi_{x_0}^{[m]} , \quad \mu^{[m]} = \pi_{x_\tau}^{[m]} \]

\[
\frac{1}{k_B} \langle \Sigma \rangle \geq D_{KL} \left[ \mu_0(m)p(m \to m') \left\| \mu_\tau(m')p(m'^\dagger \to m^\dagger) \right. \right]
\]

\[
= \Delta H(M_t) + \sum_{m,m' \in M} \mu_0(m)p(m \to m') \ln \frac{p(m \to m')}{p(m'^\dagger \to m^\dagger)}
\]

where \( \{p(m \to m')\}_{m,m'} \) are the actual transition probabilities between memory states of the computer!
Nearly-deterministic computations

Allow a maximal error rate of $\epsilon$ in the implementation of an intended deterministic computation $C : \mathcal{M} \rightarrow \mathcal{M}$ for any possible memory input $m \in \mathcal{M}$. For any such reliable computation:

$$p(m \rightarrow m') \begin{cases} 
\geq 1 - \epsilon & \text{if } m' = C(m) \\
\leq \epsilon & \text{if } m' \neq C(m) 
\end{cases}.$$
Nearly-deterministic computations

Allow a maximal error rate of $\epsilon$ in the implementation of an intended deterministic computation $\mathcal{C} : \mathcal{M} \rightarrow \mathcal{M}$ for any possible memory input $m \in \mathcal{M}$. For any such reliable computation:

$$p(m \rightarrow m') \begin{cases} 
\geq 1 - \epsilon & \text{if } m' = \mathcal{C}(m) \\
\leq \epsilon & \text{if } m' \neq \mathcal{C}(m)
\end{cases}.$$  

$$\frac{1}{k_B} \langle \Sigma \rangle \geq \Delta H(M_t) + \sum_{m,m' \in \mathcal{M}} \mu_0(m)p(m \rightarrow m') \ln \frac{p(m \rightarrow m')}{p(m'^\dagger \rightarrow m'^\dagger)}$$
The four cases

\[ p(m \rightarrow m') = \begin{cases} 
1 - \epsilon_m \geq 1 - \epsilon & \text{if } m' = C(m) \\
\epsilon_{m \rightarrow m'} \leq \epsilon & \text{if } m' \neq C(m) 
\end{cases} \]

Let \( d(m, m') \equiv p(m \rightarrow m') \ln \frac{p(m \rightarrow m')}{p(m' \rightarrow m')} \)

\begin{enumerate}
\item \( C(m) = m' \); \( C(m'^\dagger) = m'^\dagger \):

\[ -\epsilon \leq d^{(1)}(m, m') \leq \epsilon + \frac{1}{2} \epsilon^2 + O(\epsilon^3) . \]

\item \( C(m) = m' \); \( C(m'^\dagger) \neq m'^\dagger \):

\[ \ln(\epsilon^{-1}) \lesssim d^{(2)}(m, m') \leq \ln(\epsilon_{m' \rightarrow m'^\dagger}^{-1}) . \]

\item \( C(m) \neq m' \); \( C(m'^\dagger) = m'^\dagger \):

\[ \epsilon_{m \rightarrow m'} \ln \epsilon_{m \rightarrow m'} < d^{(3)}(m, m') < 0 . \]

\item \( C(m) \neq m' \); \( C(m'^\dagger) \neq m'^\dagger \):

\[ -\epsilon/e \leq d^{(4)}(m, m') = \epsilon_{m \rightarrow m'} \ln \left( \frac{\epsilon_{m \rightarrow m'}}{\epsilon_{m'^\dagger \rightarrow m'^\dagger}} \right) . \]
\end{enumerate}
The four cases

\[ p(m \to m') = \begin{cases} 
1 - \epsilon_m \geq 1 - \epsilon & \text{if } m' = C(m) \\ 
\epsilon_{m \to m'} \leq \epsilon_m \leq \epsilon & \text{if } m' \neq C(m) 
\end{cases} \]

Let \( d(m, m') \equiv p(m \to m') \ln \frac{p(m \to m')}{p(m' \to m')}. \)

1. \( C(m) = m'; C(m'^\dagger) = m'^\dagger: \)

\[ -\epsilon \leq d^{(1)}(m, m') \leq \epsilon + \frac{1}{2} \epsilon^2 + O(\epsilon^3). \]

2. \( C(m) = m'; C(m'^\dagger) \neq m'^\dagger: \)

\[
\ln(\epsilon^{-1}) \leq d^{(2)}(m, m') \leq \ln(\epsilon_{m'^\dagger \to m'^\dagger}).
\]

3. \( C(m) \neq m'; C(m'^\dagger) = m'^\dagger: \)

\[ \epsilon_{m \to m'} \ln \epsilon_{m \to m'} < d^{(3)}(m, m') < 0. \]

4. \( C(m) \neq m'; C(m'^\dagger) \neq m'^\dagger: \)

\[ -\epsilon/e \leq d^{(4)}(m, m') = \epsilon_{m \to m'} \ln \left( \frac{\epsilon_{m \to m'}}{\epsilon_{m'^\dagger \to m'^\dagger}} \right). \]
For nonreciprocated transitions:
dissipation must *diverge* as $\epsilon \to 0$!
Main result (assuming a single thermal bath: $W_{\text{diss}} = W - \Delta F_t$)

$$\beta \langle W_{\text{diss}}^{(t-sym)} \rangle + \mathcal{O}(\epsilon \ln \epsilon) > \Delta H(M_t) + \ln(1/\epsilon) \sum_{m \neq C(C(m)^\dagger)^\dagger} \mu_0(m)$$


Alternatively
(if memory states have equal local-equilibrium free energy):

$$\langle W_{\text{min}}^{(t-sym)} \rangle + \mathcal{O}(k_B T \epsilon \ln \epsilon) = k_B T \ln(1/\epsilon) \sum_{m \neq C(C(m)^\dagger)^\dagger} \mu_0(m)$$

For nonreciprocated transitions:
significant work is necessary to ensure reliability
Reciprocity ≠ logical reversibility

Minimal Work \( \langle W_{\text{min}}^{t-\text{sym}} \rangle \approx k_B T \ln(1/\epsilon) \sum_{m \neq C(C(m)^{\dagger})^{\dagger}} \mu_0(m) \)
Reciprocity ≠ logical reversibility

Mineral Work $\langle W_{\text{min}}^{t\text{-sym}} \rangle \approx k_B T \ln(1/\epsilon) \sum_{m \not\in C(C(m))} \mu_0(m)$

(assuming $m^\dagger = m$)
Erasure, by any means

Exact analysis:

\[ \beta \langle W^{t\text{-sym}} \rangle \geq (\mu_0(R) - \langle \epsilon_m \rangle) \ln \frac{1 - \epsilon_R}{\epsilon_L} \]

Low-\(\epsilon\) analysis:

\[ \sum_{m \neq \mathcal{C}_{\text{erase}}(\mathcal{C}_{\text{erase}}(m))} \mu_0(m) = \mu_0(R) \]

\[ \implies \langle W_{\text{min}}^{t\text{-sym}} \rangle \approx \mu_0(R) \ln(1/\epsilon) k_B T \]
Erasure via Arrhenius rate dynamics

$$E(M = 1, t) - E(M = 0, t) = k_B T :$$

$$E(S = 0, t) - k_B T :$$

Barrier height

Tilt

$$\Pr_{x_0, t}(M_t = R) = \langle \epsilon_m \rangle :$$
Erasure via underdamped Langevin dynamics

$$\mathcal{M}_0 = R \quad , \quad \mathcal{M}_\tau = L$$
The NAND computation $C_{\text{NAND}}$ is given by the mappings:

- $000 \mapsto 001$
- $010 \mapsto 011$
- $100 \mapsto 101$
- $111 \mapsto 110$

- $001 \mapsto 001$
- $011 \mapsto 011$
- $101 \mapsto 101$
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The NAND computation $C_{\text{NAND}}$ is given by the mappings:

$$
\begin{align*}
000 &\mapsto 001, & 001 &\mapsto 001, \\
010 &\mapsto 011, & 011 &\mapsto 011, \\
100 &\mapsto 101, & 101 &\mapsto 101, \\
111 &\mapsto 110, & 110 &\mapsto 110
\end{align*}
$$

$$
\sum_{m\neq C_{\text{NAND}}(C_{\text{NAND}}(m))} \mu_0(m) = \mu_0(m_{000}) + \mu_0(m_{010}) + \mu_0(m_{100}) + \mu_0(m_{111})
$$

$$
\implies \langle W_{\text{min}}^{t-\text{sym}} \rangle \approx k_B T \left( \mu_0(m_{000}) + \mu_0(m_{010}) + \mu_0(m_{100}) + \mu_0(m_{111}) \right) \ln(1/\epsilon)
$$
The NAND computation $\mathcal{C}_{\text{NAND}}$ is given by the mappings:

\[
\begin{align*}
000 & \mapsto 001 & 001 & \mapsto 001 \\
010 & \mapsto 011 & 011 & \mapsto 011 \\
100 & \mapsto 101 & 101 & \mapsto 101 \\
111 & \mapsto 110 & 110 & \mapsto 110
\end{align*}
\]

\[
\sum_{m \neq \mathcal{C}_{\text{NAND}}(m)} \mu_0(m) = \mu_0(m_{000}) + \mu_0(m_{010}) + \mu_0(m_{100}) + \mu_0(m_{111})
\]

\[
\implies \langle W_{\text{min}}^{t\text{-sym}} \rangle \approx k_B T (\mu_0(m_{000}) + \mu_0(m_{010}) + \mu_0(m_{100}) + \mu_0(m_{111})) \ln(1/\epsilon)
\]

vs.

\[
\langle W_{\text{min}}^{\text{Landauer}} \rangle = -k_B T \Delta H(\mathcal{M}_t) \approx k_B T H(\mathcal{M}_0^{(\text{out})} |\mathcal{M}_0^{(\text{in}_1)}, \mathcal{M}_0^{(\text{in}_2)})
\]
Is time-symmetric control relevant?

This time-symmetric signal drives all logical transformations in a computer (program, input memory, working memory, etc. are all part of memory system).

Extreme reliability $\rightarrow \sim 30k_BT$ dissipation per logic transformation

$\sim 40$ times the Landauer bound
Consequence for reliable biological functionality.

\[ \text{ATP} \rightarrow \text{ADP} + P_i \quad \text{yields} \quad \sim 30 \text{ kJ} / \text{mol} \approx 12 - 25 \ k_B T \quad \text{per ATP} \]

- Biological systems in NESS
- Biological system transitioning between NESSs
## Avoiding the Error–Dissipation tradeoff

Use less reliable components with error correction?
Avoiding the Error–Dissipation tradeoff

Use less reliable components with error correction? (Doesn’t help)
Three ways to avoid the error–dissipation tradeoff

1. Engineer the time-reversal symmetries of the memory elements for special purpose computations (How? Alec Boyd’s talk, tomorrow!)

2. Use time-asymmetric control (Hidden costs of time asymmetry?)

3. Intertial computing—avoid metastability (Problematic but interesting!)
Thanks!

This talk is based on:


...and related work in progress.

Also relevant to this workshop:

Time-symmetric control of metastable memories can only achieve reliable nonreciprocated transitions via loss of stability and subsequent dissipation.

This trouble is topologically guaranteed!
Time-\textit{asymmetric} control can be used to guide the probability density reliably, while maintaining metastability.
Transition-specific results

\[ W_{\text{min}}^{t-\text{sym}}(m \rightarrow m') = F_{x_0}^{(m')} - F_{x_0}^{(m)} + k_B T \ln \frac{p(m \rightarrow m')}{p(m'^\dagger \rightarrow m'^\dagger)} \]