Effects of control restrictions in quantum thermodynamics

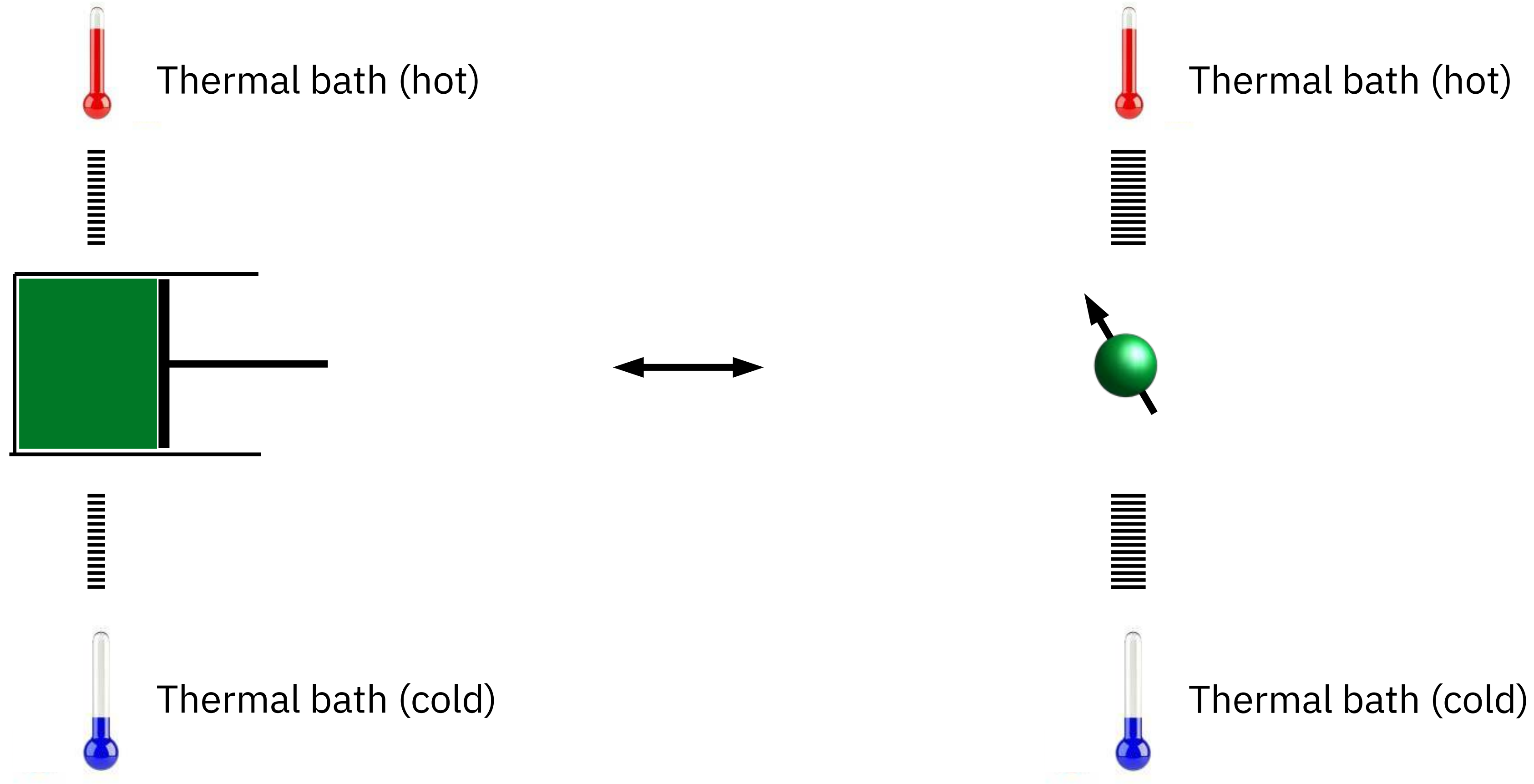
Henrik Wilming ETH Zürich
henrikw@phys.ethz.ch

Work with: Rodrigo Gallego, Jens Eisert, Jaqueline Lekscha, Martí Perarnau-Llobet, Arnau Riera

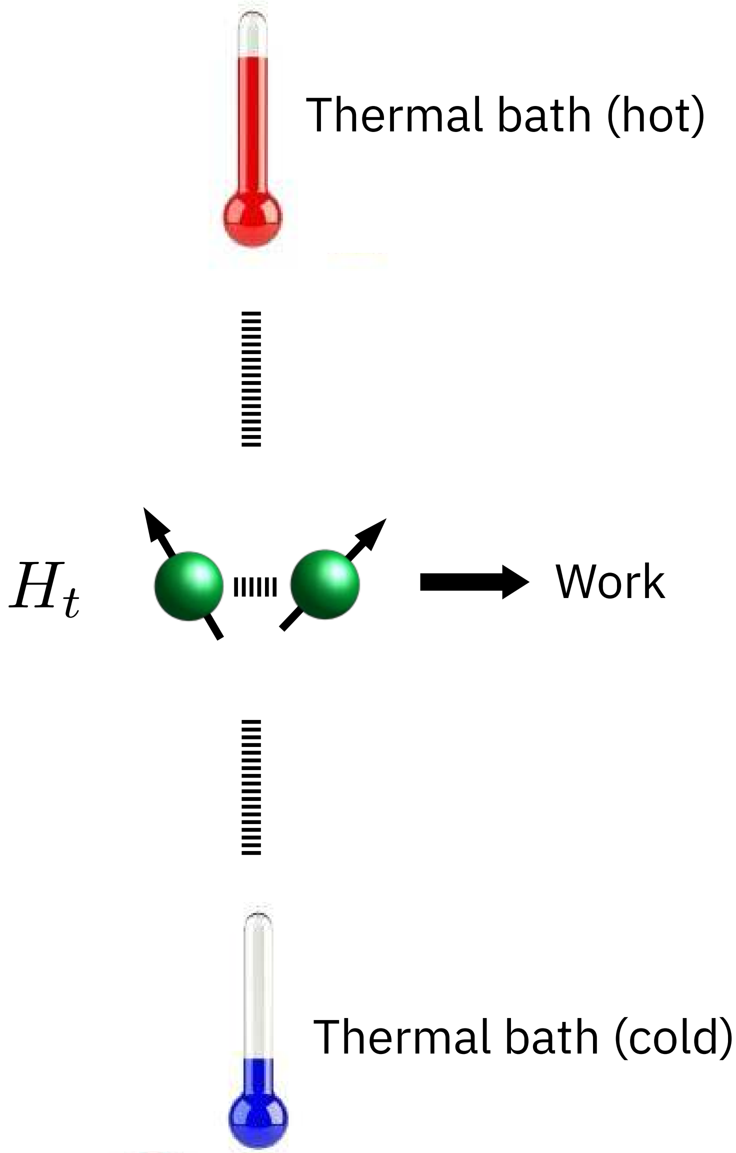
CSH Online Workshop “Stochastic thermodynamics of complex systems”, May 29th, 2020

Macroscopic machines: Only **limited control** neccessary to (approximately) **achieve thermodynamic bounds**.

Quantum machines: Also have thermodynamic bounds. Less clear what implications limited experimental control has on being able to achieve them.



The model



We can do any combination of three elementary operations:

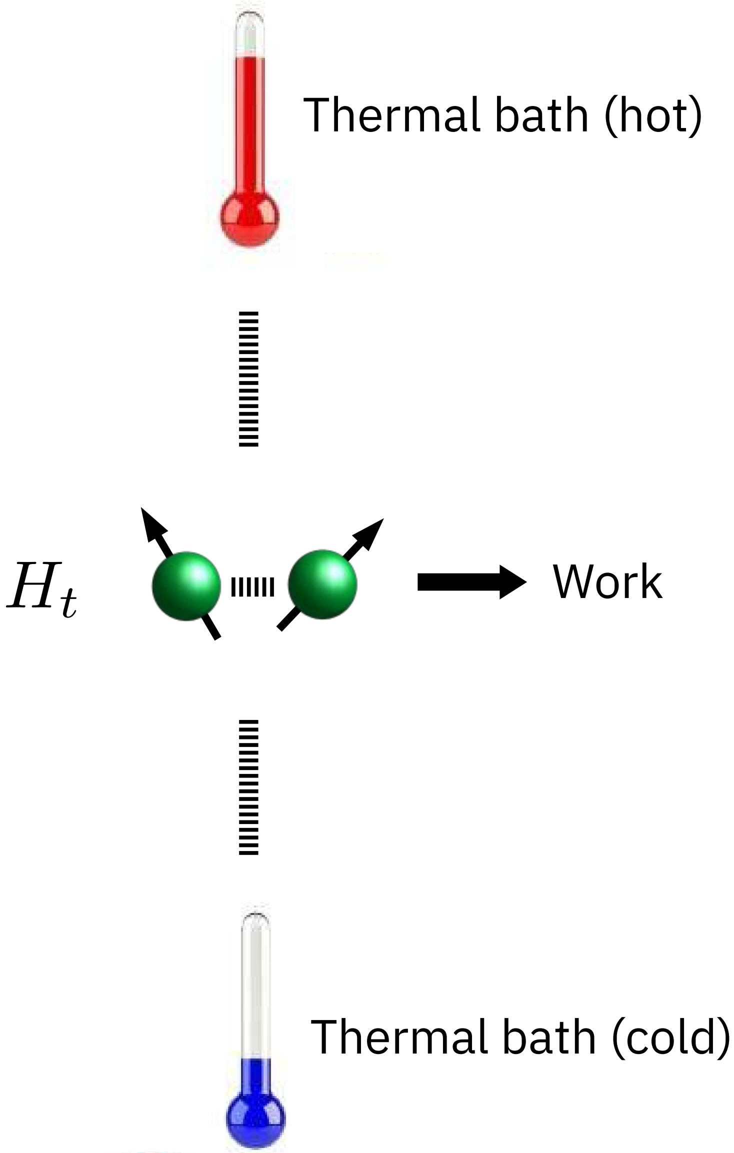
1. Change the Hamiltonian of system over time (**unitary dynamics**). Associated with **average work**:

$$W = \mathrm{Tr}\big(\rho_{t_1} H_{t_1}\big) - \mathrm{Tr}\left(U \rho_{t_1} U^\dagger H_{t_2}\right)$$

2. (De-)Couple system with **one** of the baths.

3. Let system thermalize (if coupled with bath). Associated with **heat flow**.

The model: Landscape of assumptions



Various scenarios to be considered:

i. Coupling to bath can be fixed or controllable

H. W.., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

ii. If fixed, the coupling can be weak or strong

M. Perarnau-Llobet, H. W., A. Riera, R. Gallego, and J. Eisert, *Phys. Rev. Lett. 120, 120602* (2018)

iii. Hamiltonian can be restricted or can be controlled arbitrarily

H. W., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

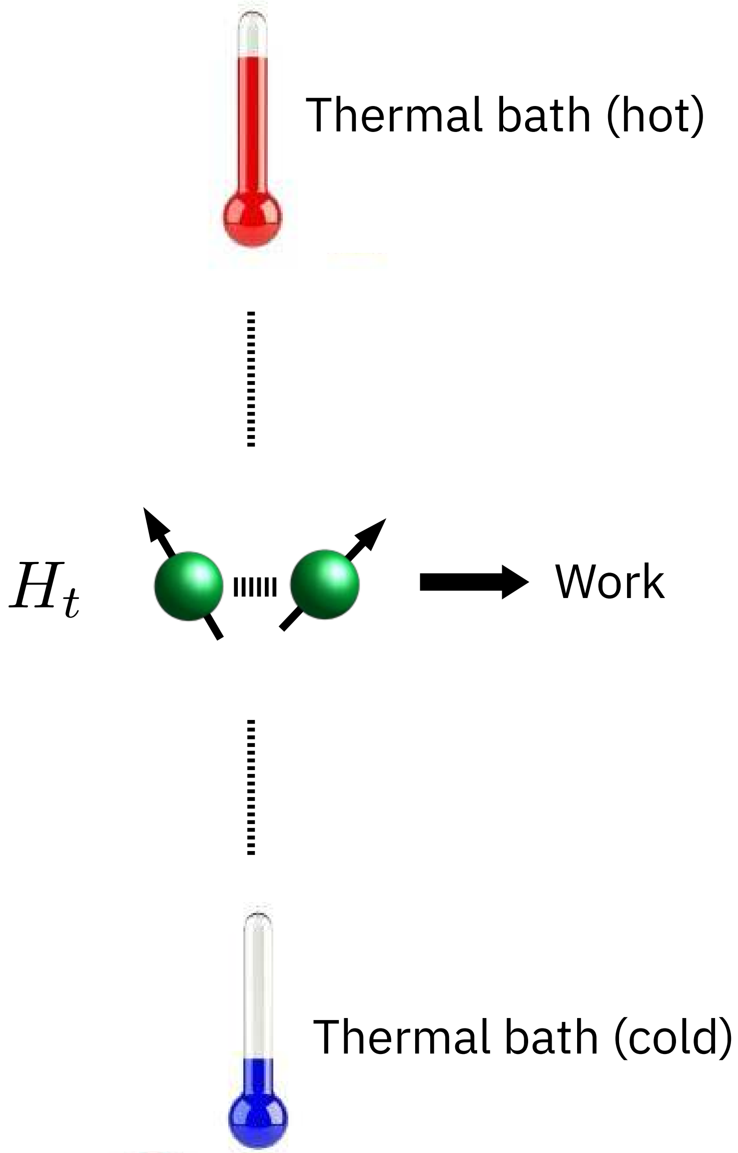
J. Lekscha, H. W., J. Eisert, and R. Gallego, *Phys. Rev. E 97 (2), 022142* (2018)

iv. System thermalizes to Gibbs state or not

M. Perarnau-Llobet, A. Riera, R. Gallego, H. W., and J. Eisert *New J. Phys. 18.12* (2016)

We have general results/bounds in all these settings.
But today I will focus on simple toy models.

Thermal machines: Weak coupling



The *weak-coupling limit* consists of:

1. Work-cost of (de)coupling the bath can be neglected.

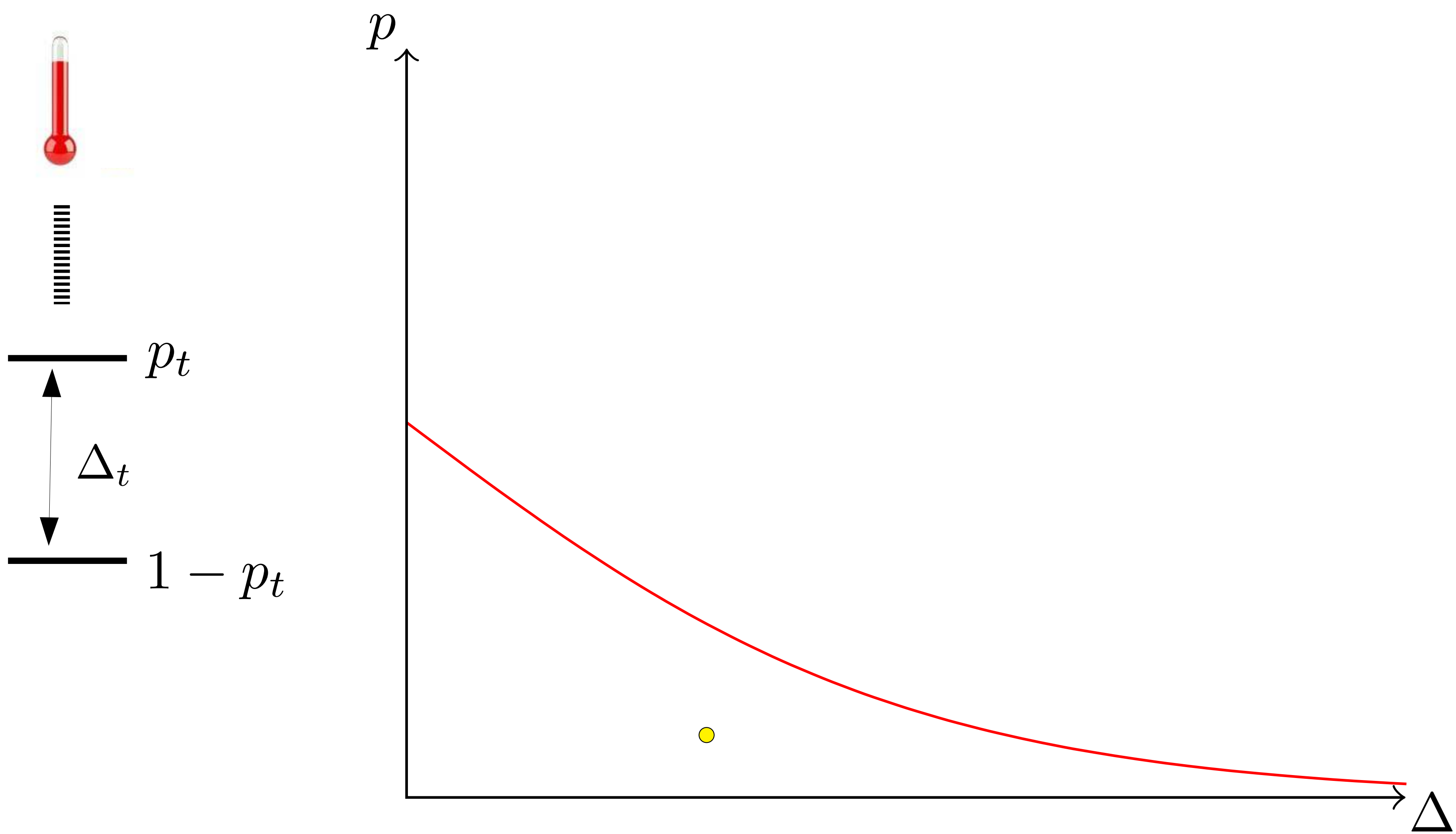
2. System thermalizes to **Gibbs state**:

$$\rho_t \mapsto \omega_\beta(H_t) = \frac{e^{-\beta H_t}}{Z_t}$$

3. Heat transferred from the bath in a thermalization step is given by

$$Q = \mathrm{Tr}(H_t \omega_\beta(H_t)) - \mathrm{Tr}(H_t \rho_t)$$

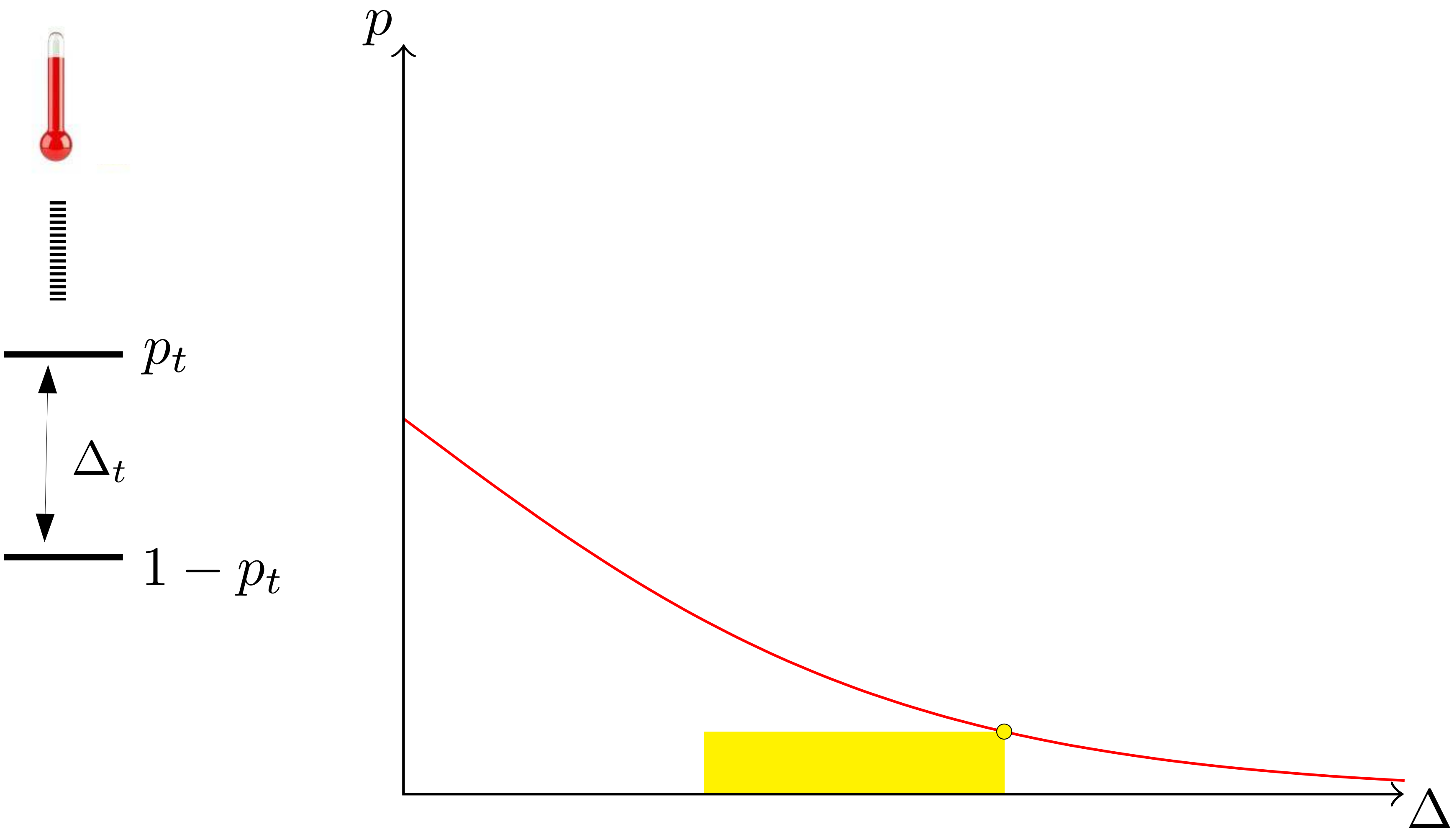
Example: Work extraction using a two-level system



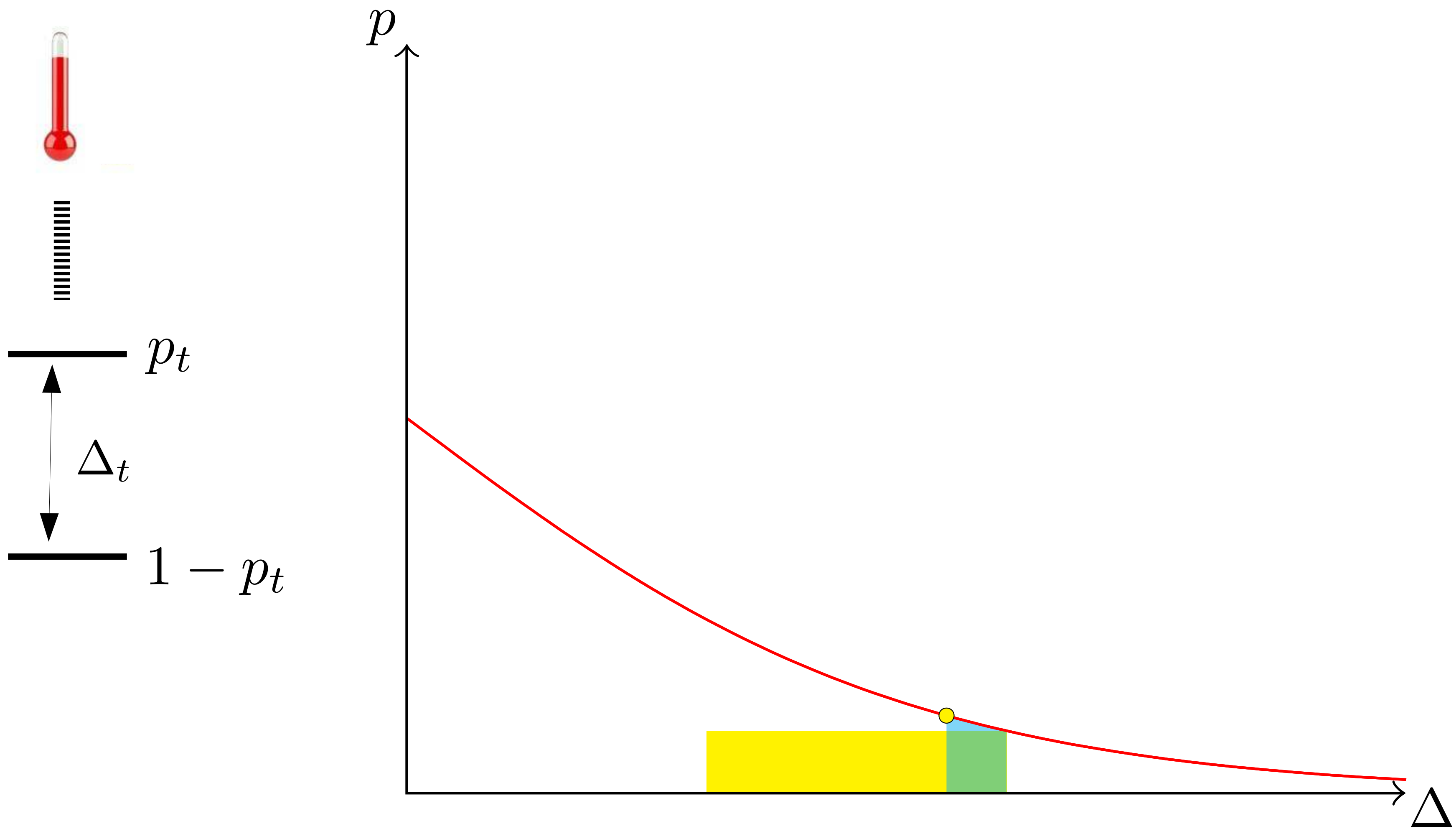
In the adiabatic parts of the protocol, **any unitary transformation** can be implemented. But one can show that it is always optimal to “stay classical” if initial state diagonal in energy eigenbasis.

H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

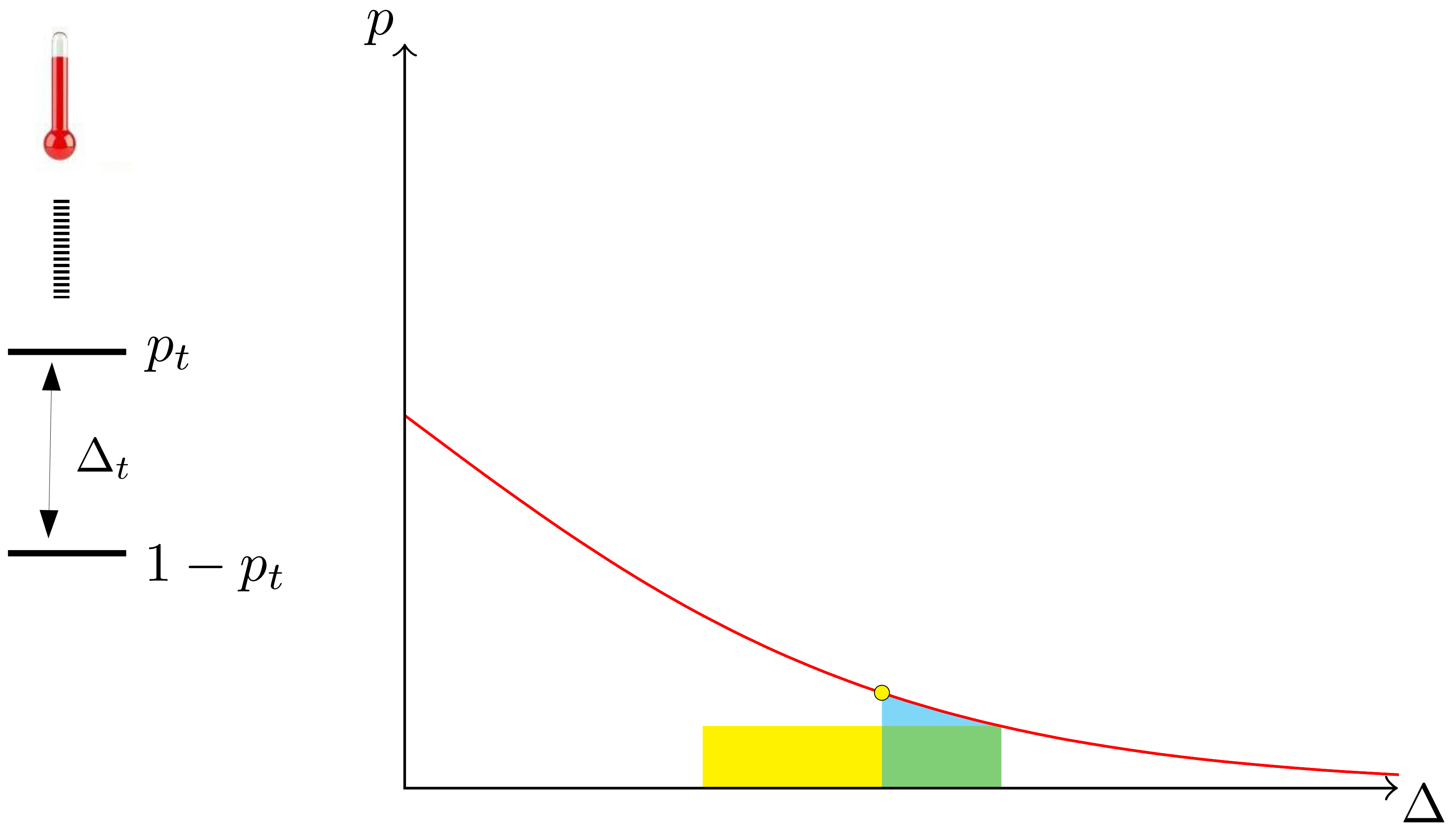
Example: Work extraction using a two-level system



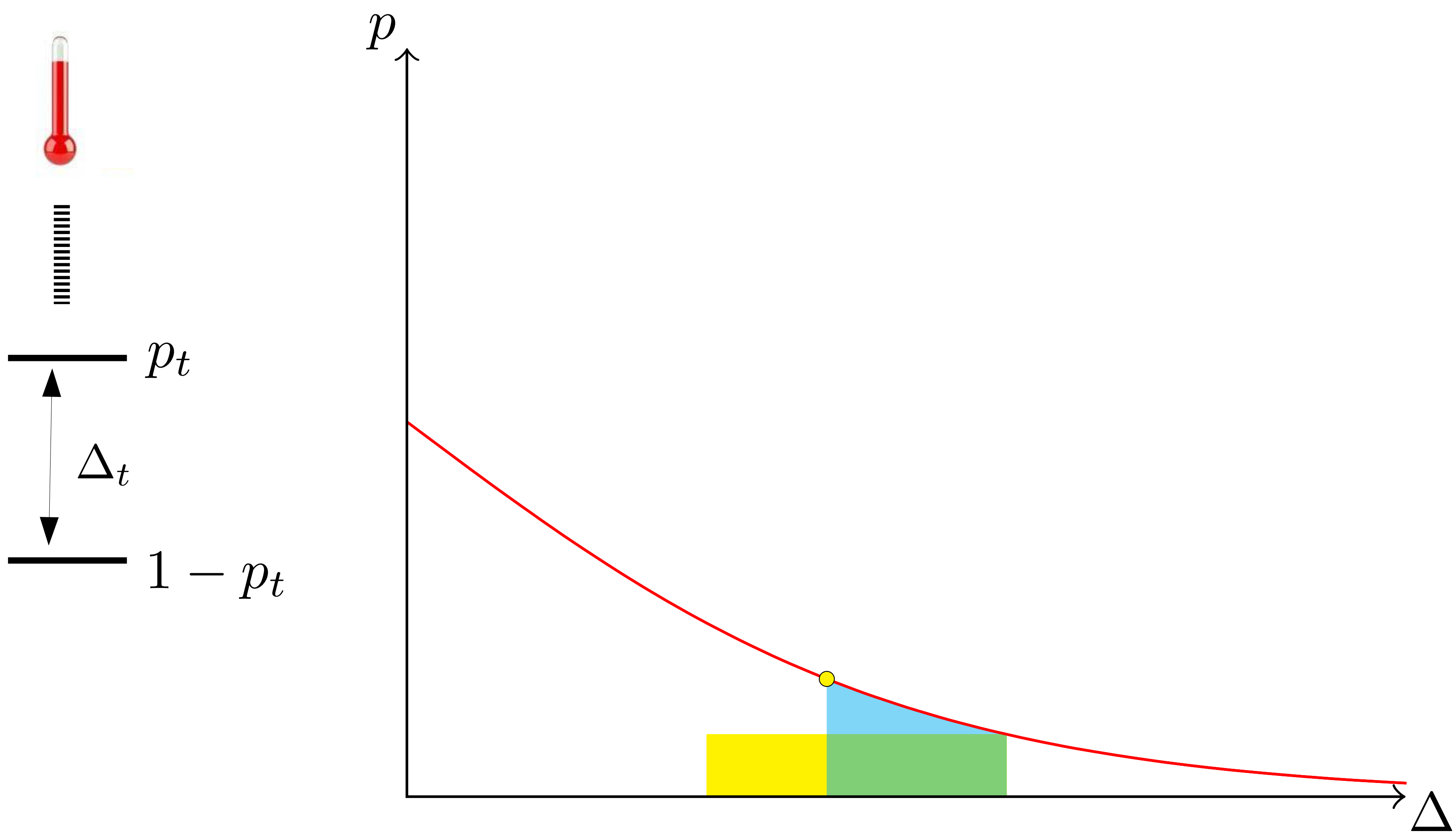
Example: Work extraction using a two-level system



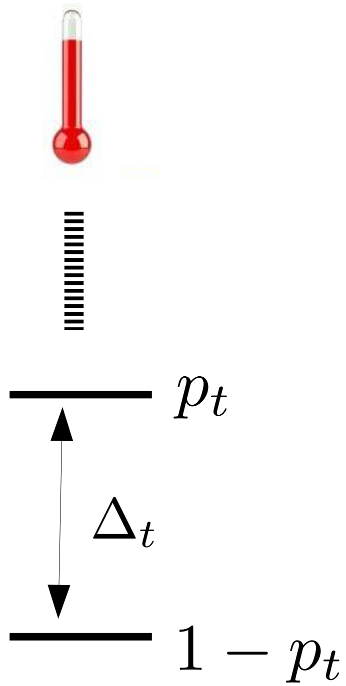
Example: Work extraction using a two-level system

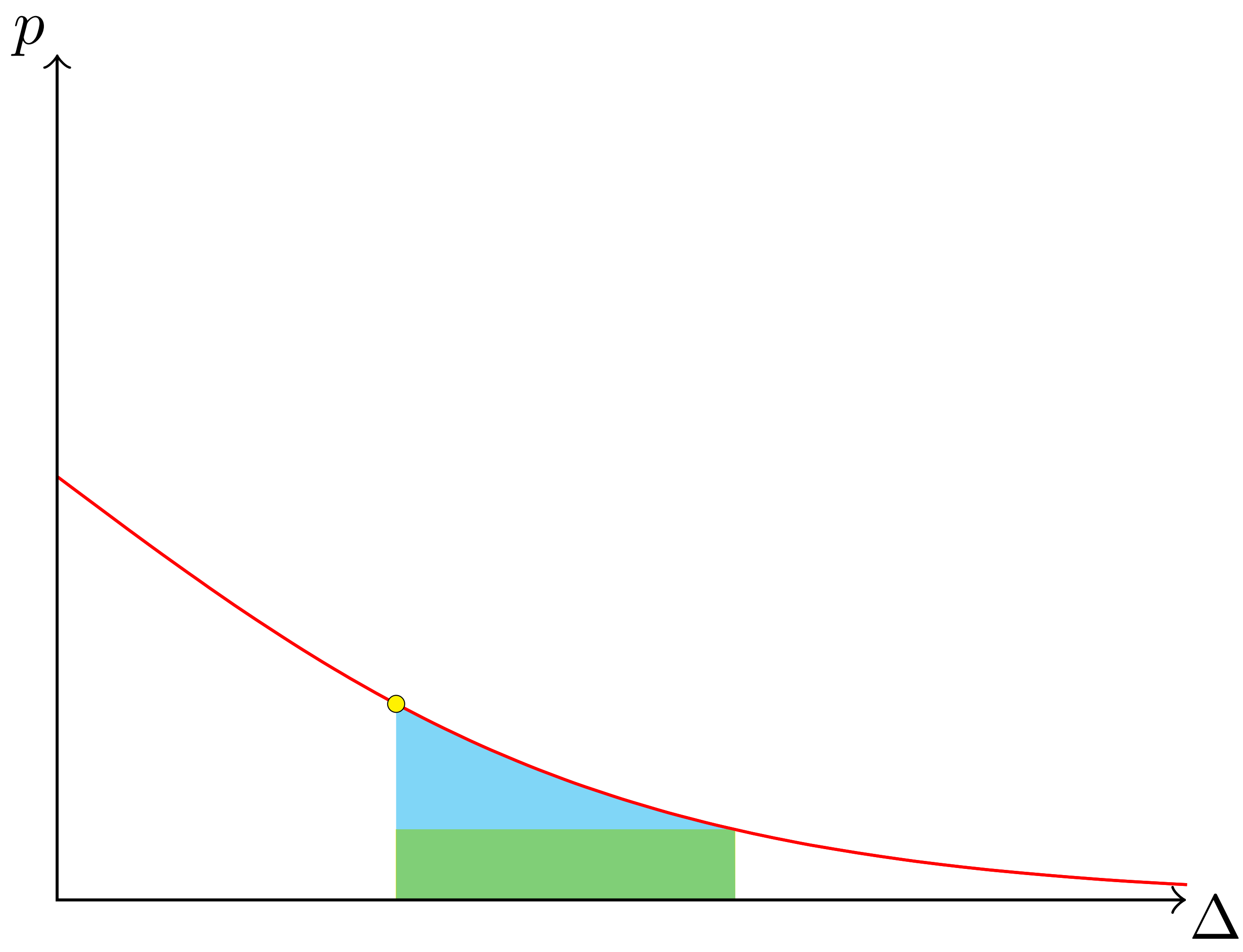


Example: Work extraction using a two-level system

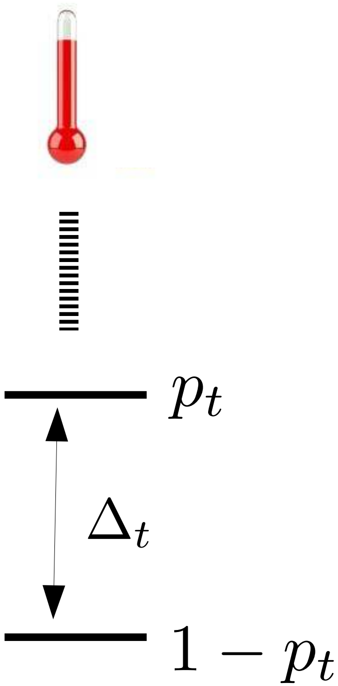


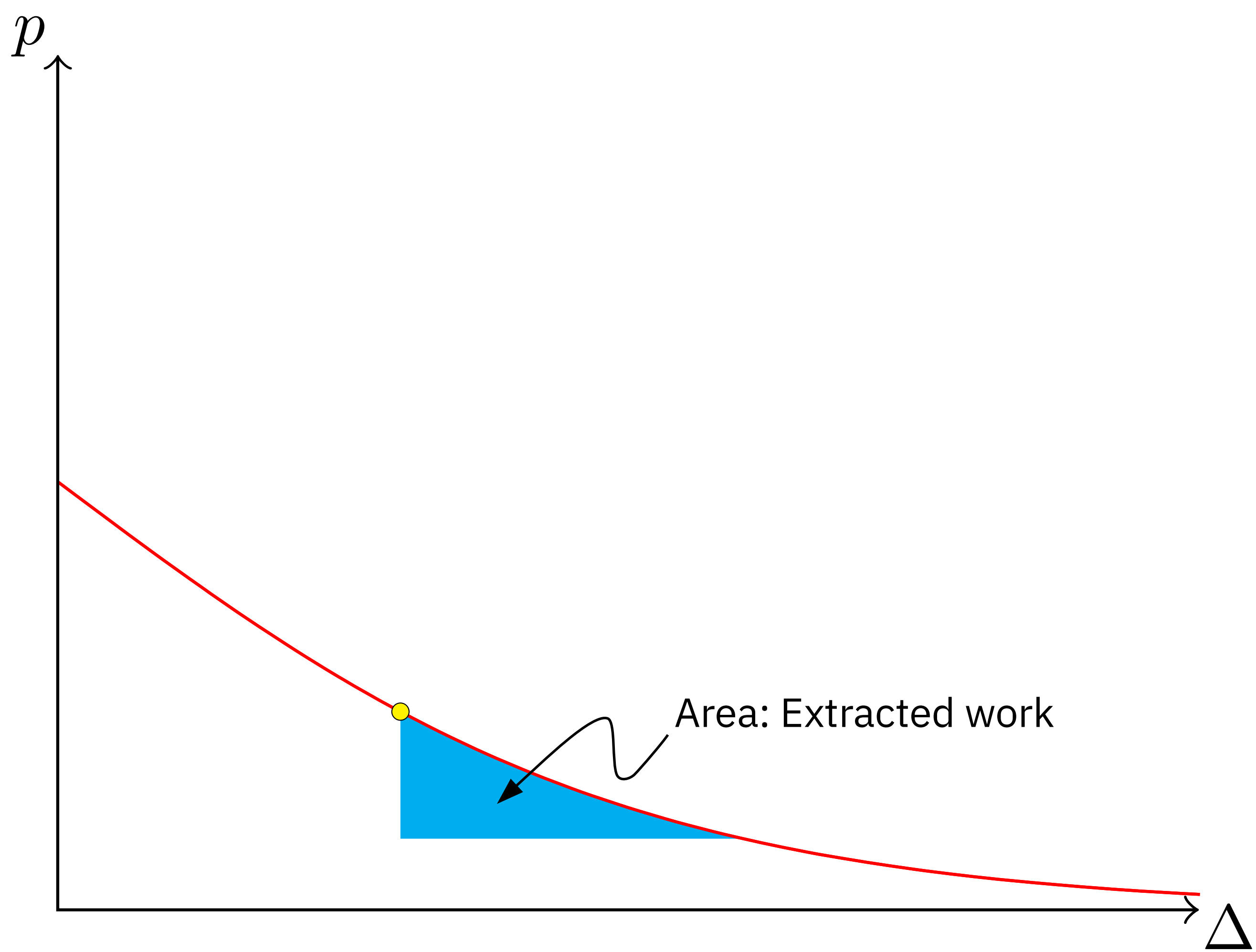
Example: Work extraction using a two-level system





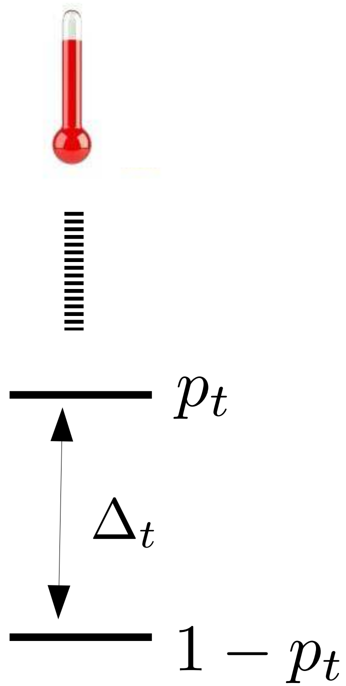
Example: Work extraction using a two-level system

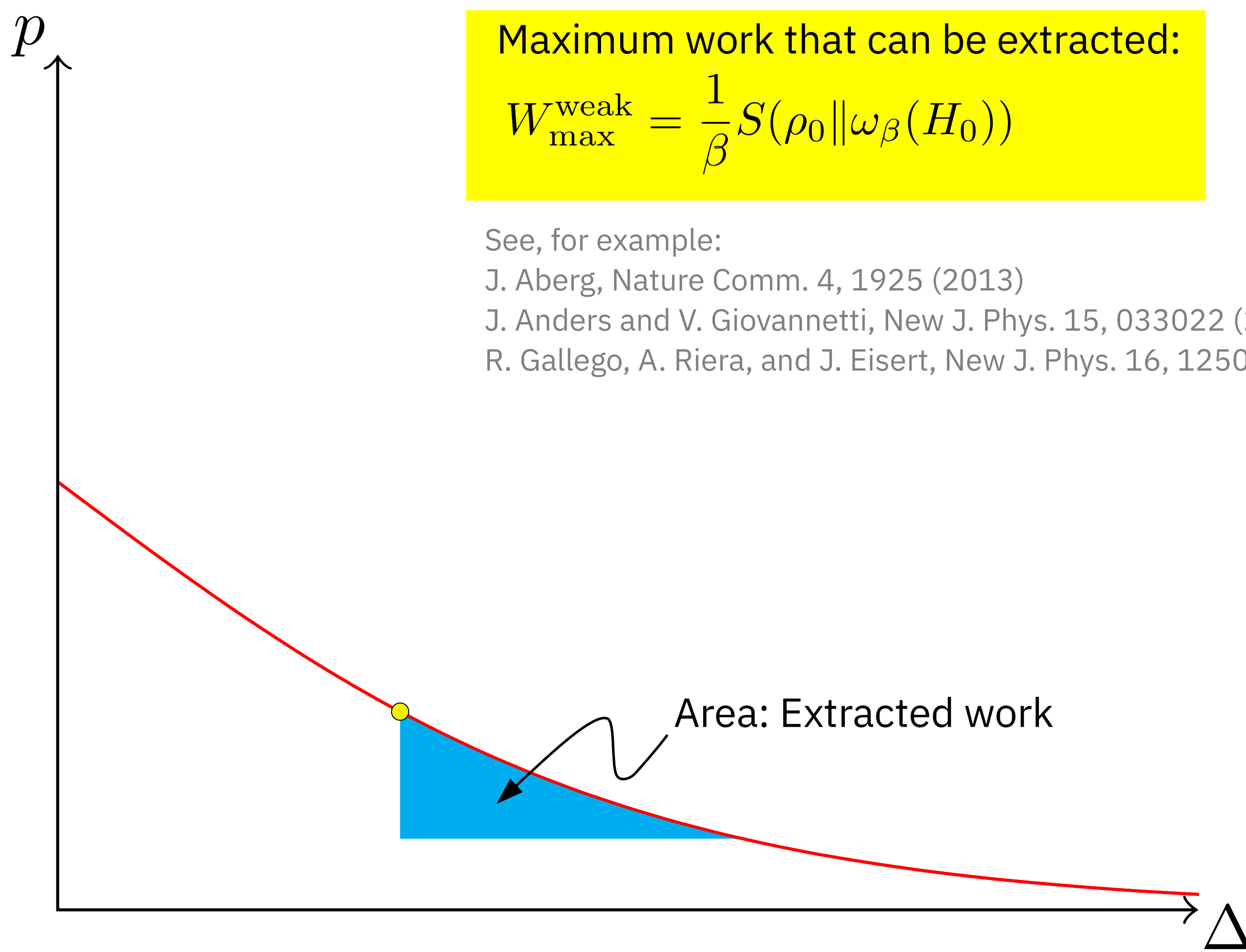




Similar protocol exists for **any** initial quantum state and **any** Hamiltonian.

Example: Work extraction using a two-level system





Maximum work that can be extracted:

$$W_{\max}^{\text{weak}} = \frac{1}{\beta} S(\rho_0 \| \omega_\beta(H_0))$$

See, for example:

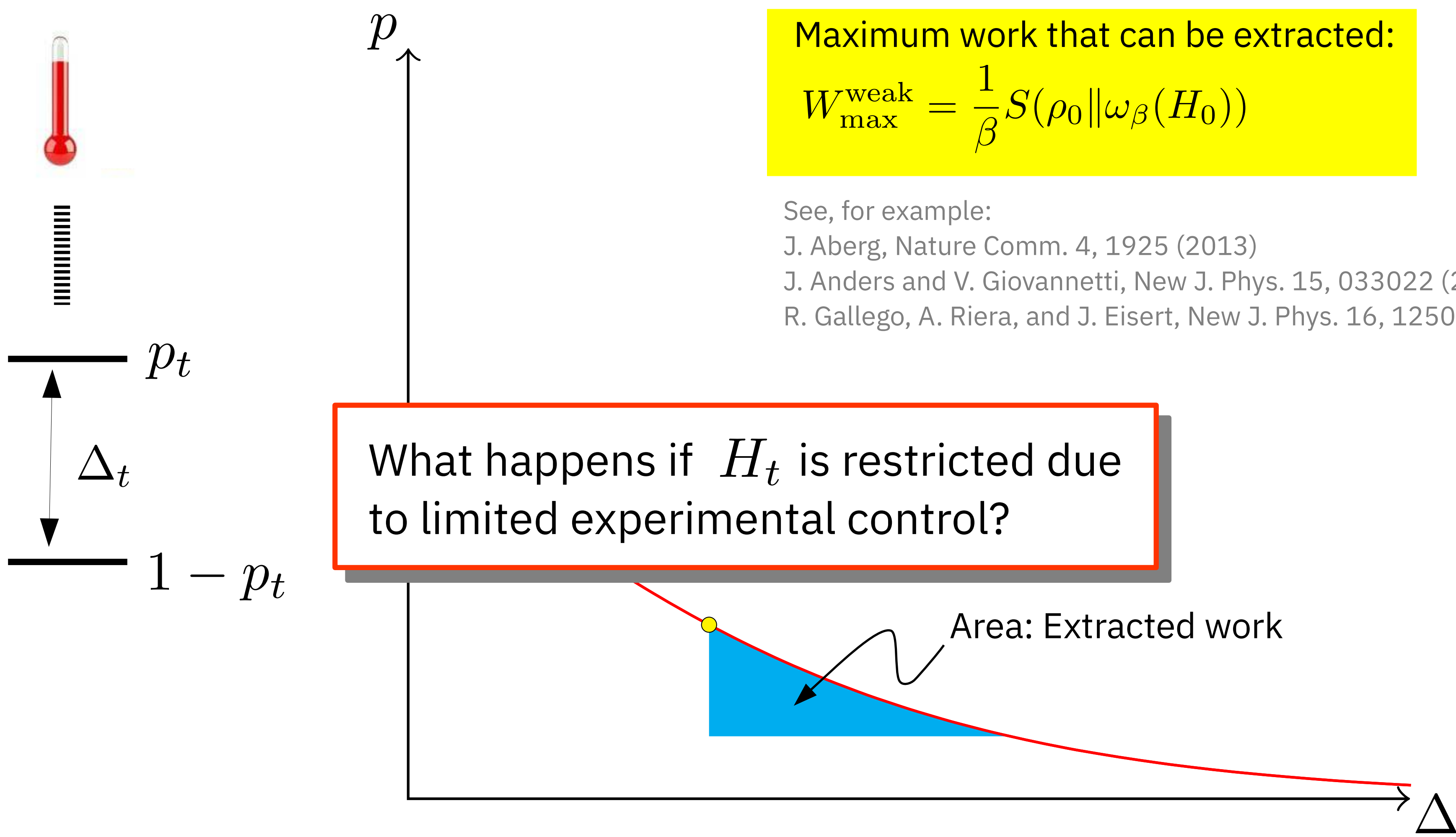
J. Aberg, Nature Comm. 4, 1925 (2013)

J. Anders and V. Giovannetti, New J. Phys. 15, 033022 (2013)

R. Gallego, A. Riera, and J. Eisert, New J. Phys. 16, 125009 (2014)

Similar protocol exists for **any** initial quantum state and **any** Hamiltonian.

Example: Work extraction using a two-level system



Maximum work that can be extracted:

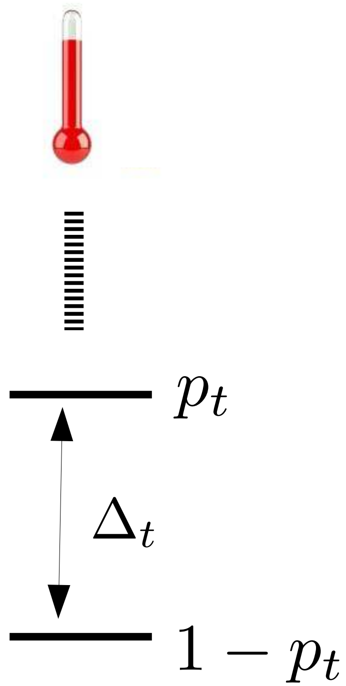
$$W_{\max}^{\text{weak}} = \frac{1}{\beta} S(\rho_0 \| \omega_\beta(H_0))$$

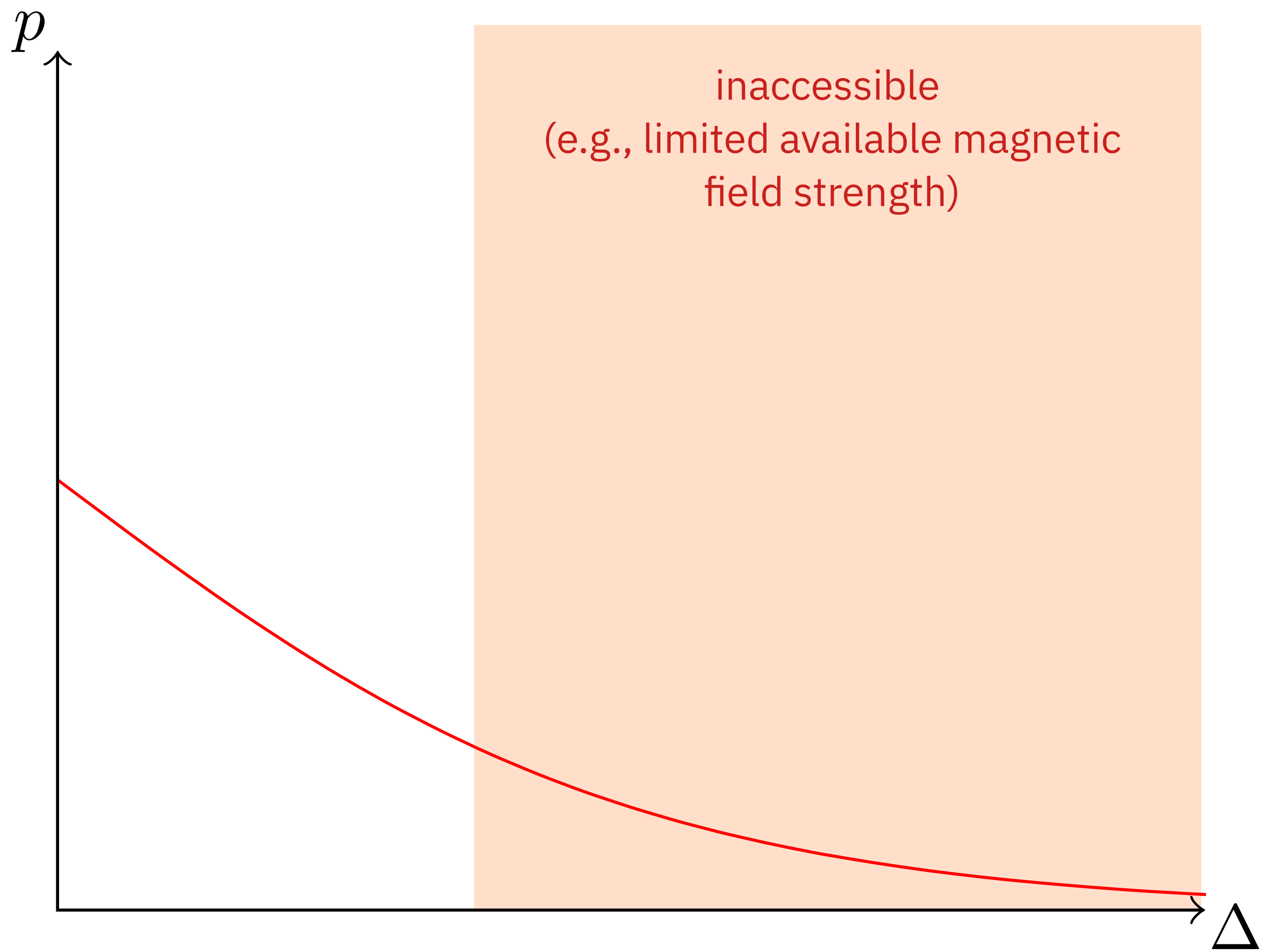
See, for example:
J. Aberg, Nature Comm. 4, 1925 (2013)
J. Anders and V. Giovannetti, New J. Phys. 15, 033022 (2013)
R. Gallego, A. Riera, and J. Eisert, New J. Phys. 16, 125009 (2014)

What happens if H_t is restricted due to limited experimental control?

Similar protocol exists for **any** initial quantum state and **any** Hamiltonian.

Toy example of limited control: Work extraction using a two-level system

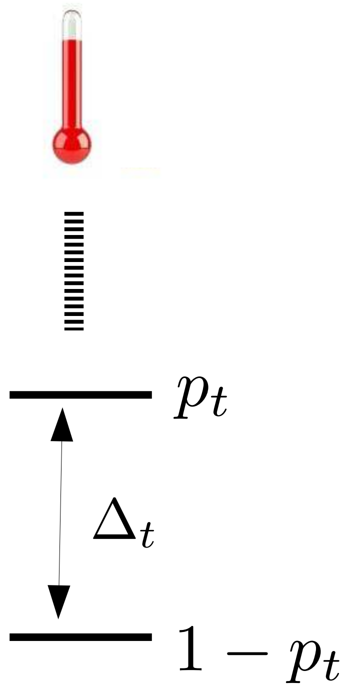


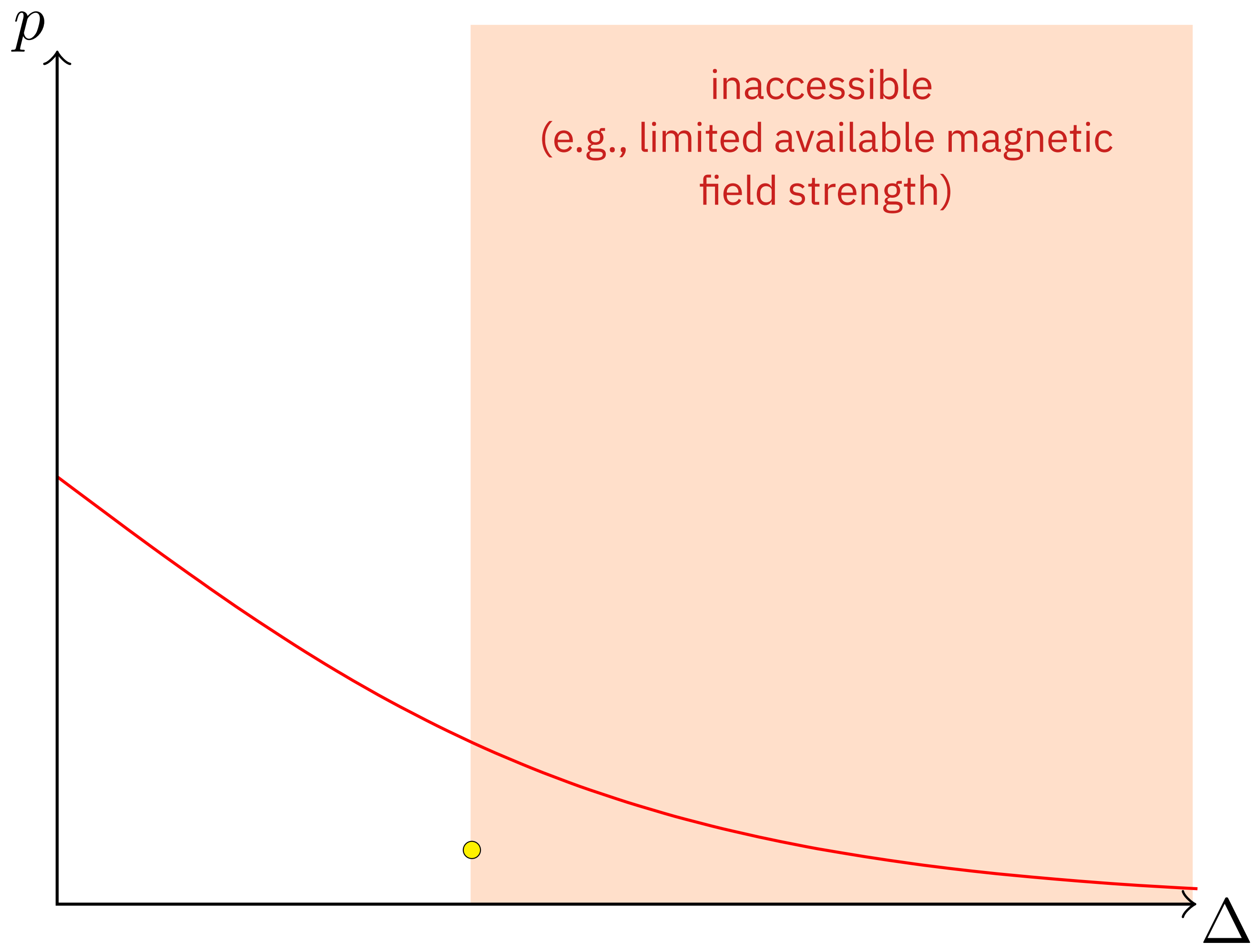


In the adiabatic parts of the protocol, **still any unitary transformation** can be implemented.

H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

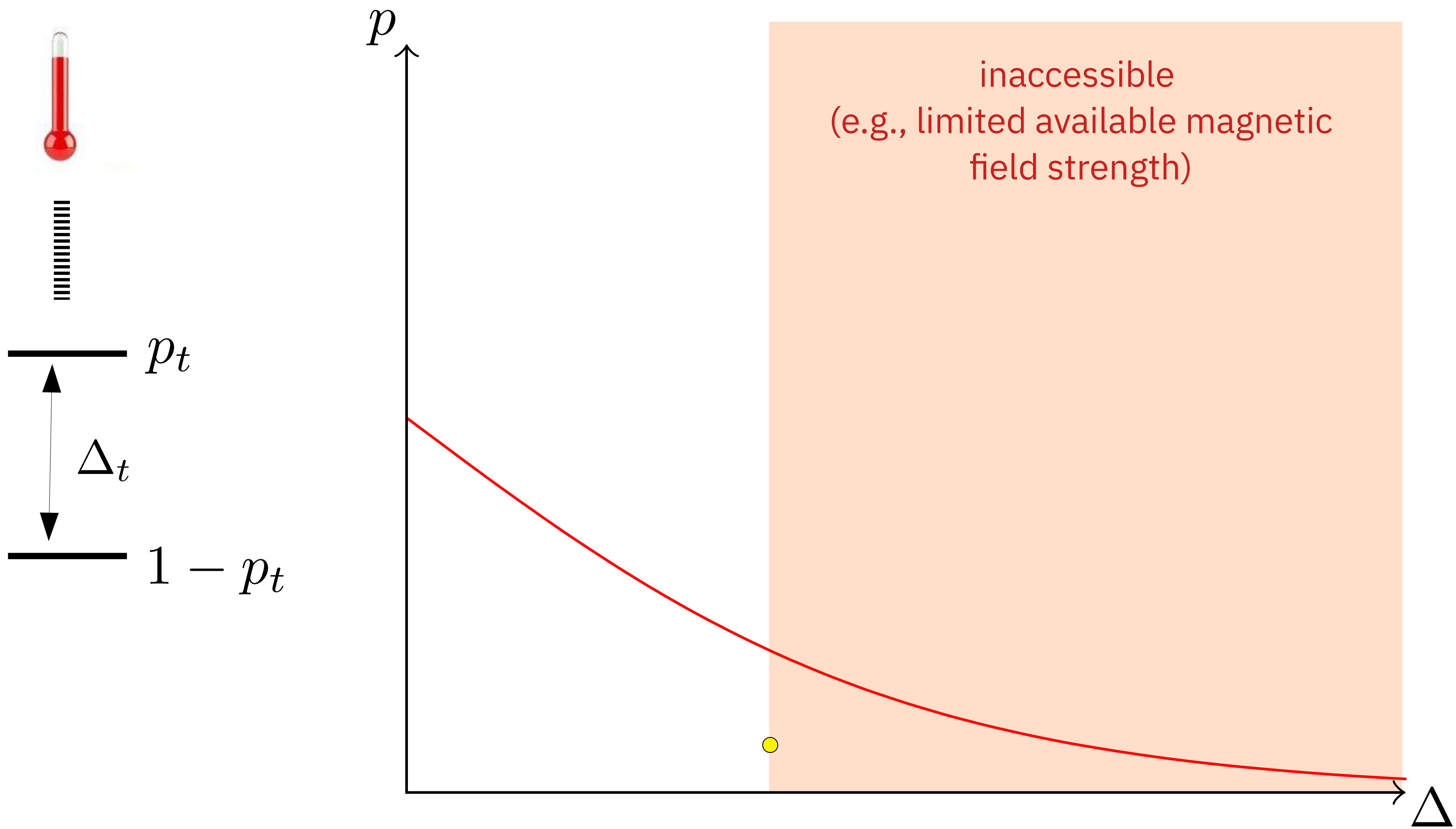
Toy example of limited control: Work extraction using a two-level system





H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

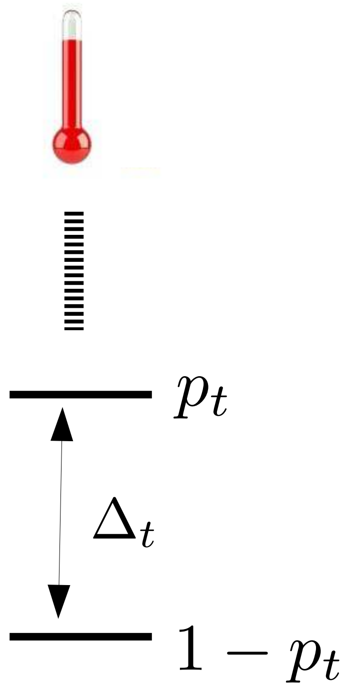
Toy example of limited control: Work extraction using a two-level system

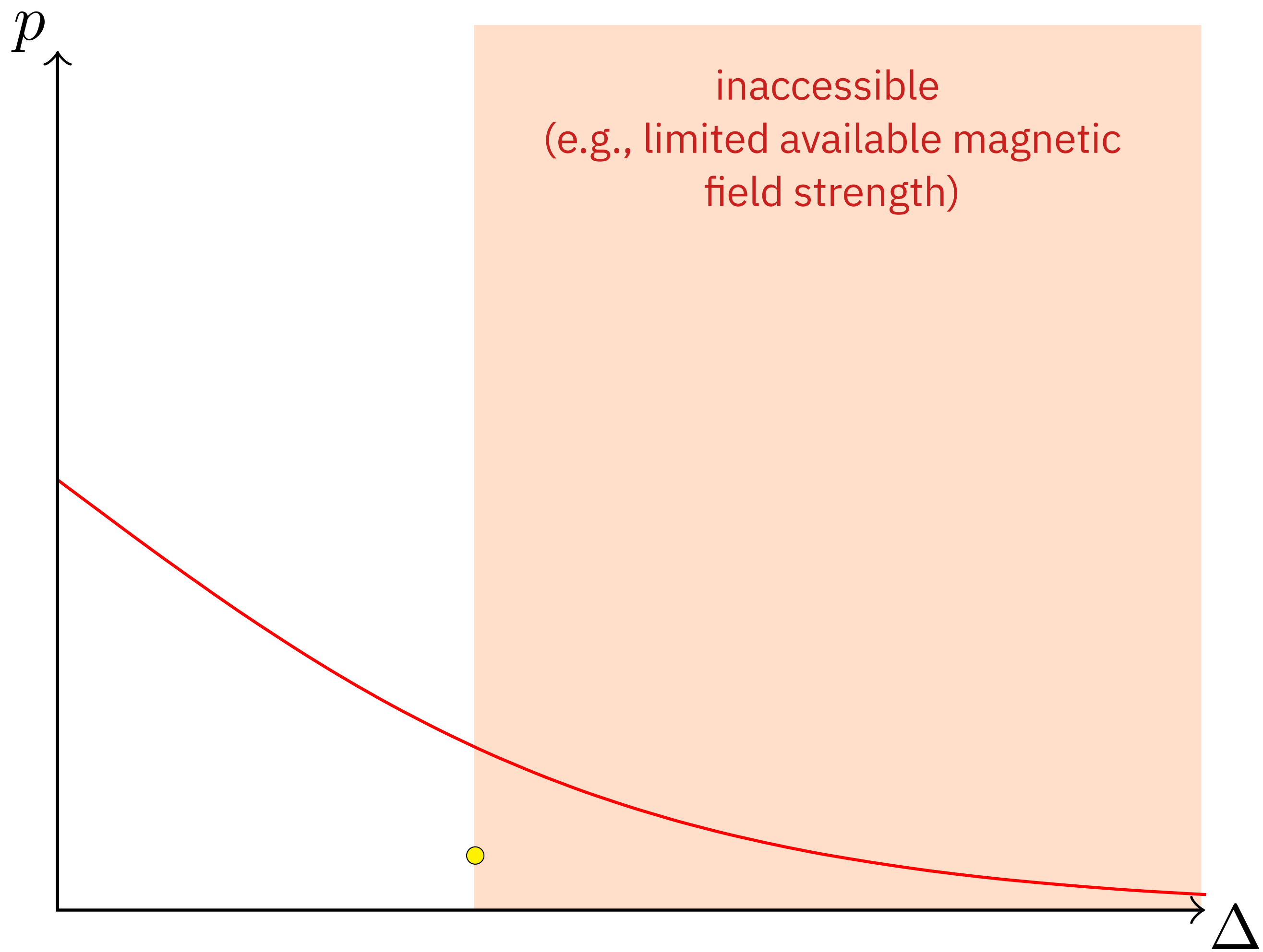


Cannot perform counter-clockwise **cycle:**
No positive work can be extracted by any protocol.

H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

Toy example of limited control: Work extraction using a two-level system



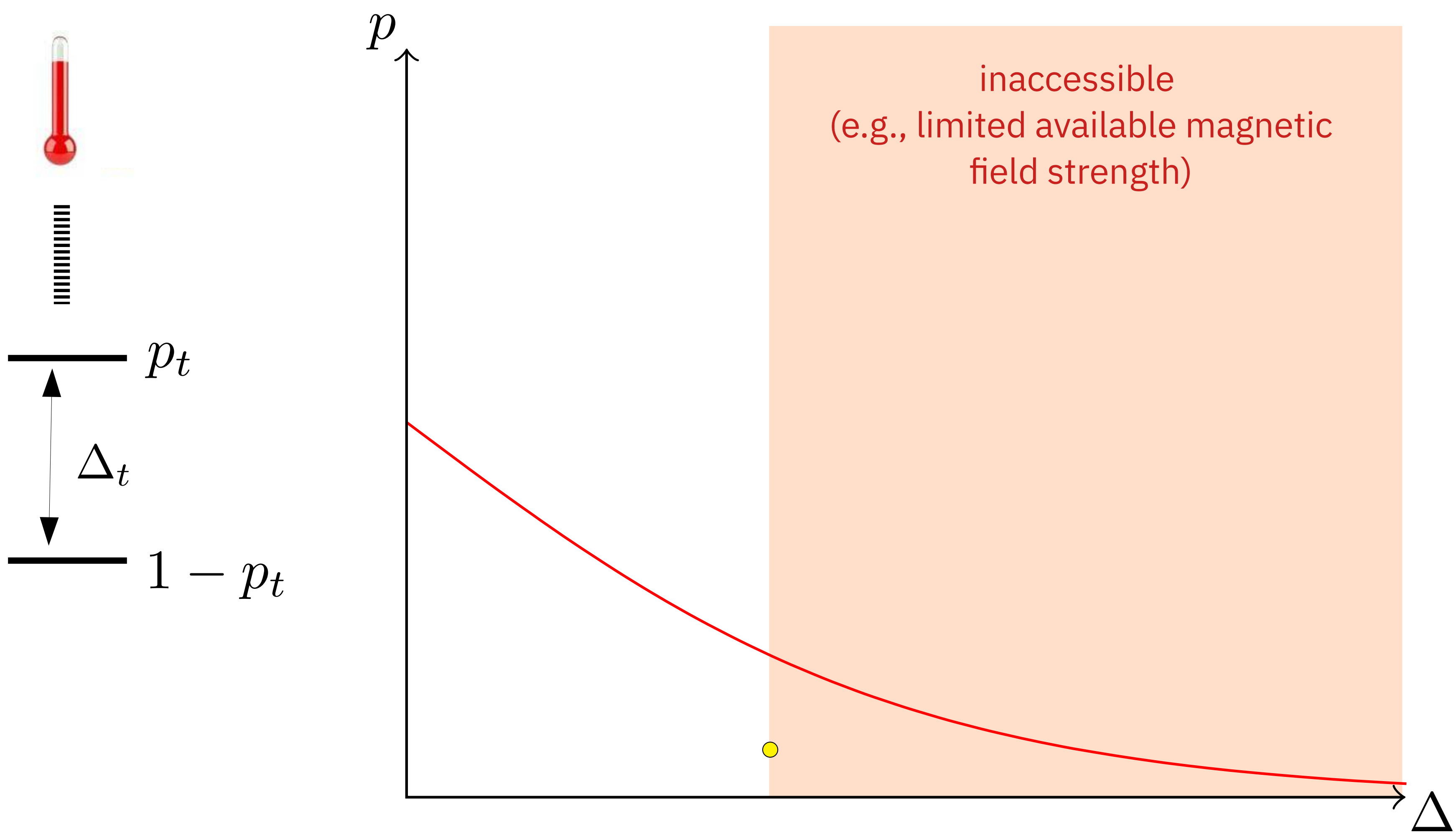


Initial configuration is "**passive**" due to limited Hamiltonian control.

Cannot perform counter-clockwise **cycle**:
No positive work can be extracted by any protocol.

H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

Toy example of limited control: Work extraction using a two-level system



Initial configuration is "**passive**" due to limited Hamiltonian control.

Cannot perform counter-clockwise **cycle:**
No positive work can be extracted by any protocol.

Reason: Cannot cross isothermal (only way to move vertically is by thermalization to the isothermal).

H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

Toy example: The role of control over the bath and the **β-swap**

So far, bath simply thermalizes. What if we had some control over the bath?

For any temperature and qubit Hamiltonian, there exists a (quantum) channel $\mathcal{G}_\beta$ s.t.:

H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).
Á. M. Alhambra, M. Lostaglio, and C. Perry, Quantum 3, 188 (2019).
K. Korzekwa, M. Lostaglio, arXiv: 2005.02403.

Toy example: The role of control over the bath and the **β-swap**

So far, bath simply thermalizes. What if we had some control over the bath?

For any temperature and qubit Hamiltonian, there exists a (quantum) channel $\mathcal{G}_\beta$ s.t.:

1) It can be implemented by coupling the qubit to a **single thermal harmonic oscillator** or (approx.) to a **bath consisting of few qubits** using an energy-preserving unitary.

H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).
Á. M. Alhambra, M. Lostaglio, and C. Perry, Quantum 3, 188 (2019).
K. Korzekwa, M. Lostaglio, arXiv: 2005.02403.

Toy example: The role of control over the bath and the **β-swap**

So far, bath simply thermalizes. What if we had some control over the bath?

For any temperature and qubit Hamiltonian, there exists a (quantum) channel $\mathcal{G}_\beta$ s.t.:

1) It can be implemented by coupling the qubit to a **single thermal harmonic oscillator** or (approx.) to a **bath consisting of few qubits** using an energy-preserving unitary.

2) It preserves the thermal state:

$$\mathcal{G}_\beta[\omega_\beta(H)] = \omega_\beta(H)$$

H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).
Á. M. Alhambra, M. Lostaglio, and C. Perry, Quantum 3, 188 (2019).
K. Korzekwa, M. Lostaglio, arXiv: 2005.02403.

Toy example: The role of control over the bath and the **β-swap**

So far, bath simply thermalizes. What if we had some control over the bath?

For any temperature and qubit Hamiltonian, there exists a (quantum) channel $\mathcal{G}_\beta$ s.t.:

1) It can be implemented by coupling the qubit to a **single thermal harmonic oscillator** or (approx.) to a **bath consisting of few qubits** using an energy-preserving unitary.

2) It preserves the thermal state:

$$\mathcal{G}_\beta[\omega_\beta(H)] = \omega_\beta(H)$$

Suffices to ensure that 2nd law cannot be violated in protocols using such thermalization maps

H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).
Á. M. Alhambra, M. Lostaglio, and C. Perry, Quantum 3, 188 (2019).
K. Korzekwa, M. Lostaglio, arXiv: 2005.02403.

Toy example: The role of control over the bath and the **β-swap**

So far, bath simply thermalizes. What if we had some control over the bath?

For any temperature and qubit Hamiltonian, there exists a (quantum) channel $\mathcal{G}_\beta$ s.t.:

1) It can be implemented by coupling the qubit to a **single thermal harmonic oscillator** or (approx.) to a **bath consisting of few qubits** using an energy-preserving unitary.

2) It preserves the thermal state:

$$\mathcal{G}_\beta[\omega_\beta(H)] = \omega_\beta(H)$$

Suffices to ensure that 2nd law cannot be violated in protocols using such thermalization maps

3) It induces an **approximate population inversion**:

$$\mathcal{G}_\beta[|0\rangle\langle 0|] = (1 - \mathrm{e}^{-\beta\Delta})|0\rangle\langle 0| + \mathrm{e}^{-\beta\Delta}|1\rangle\langle 1|$$

H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).
Á. M. Alhambra, M. Lostaglio, and C. Perry, Quantum 3, 188 (2019).
K. Korzekwa, M. Lostaglio, arXiv: 2005.02403.

Toy example: The role of control over the bath and the **β-swap**

So far, bath simply thermalizes. What if we had some control over the bath?

For any temperature and qubit Hamiltonian, there exists a (quantum) channel $\mathcal{G}_\beta$ s.t.:

1) It can be implemented by coupling the qubit to a **single thermal harmonic oscillator** or (approx.) to a **bath consisting of few qubits** using an energy-preserving unitary.

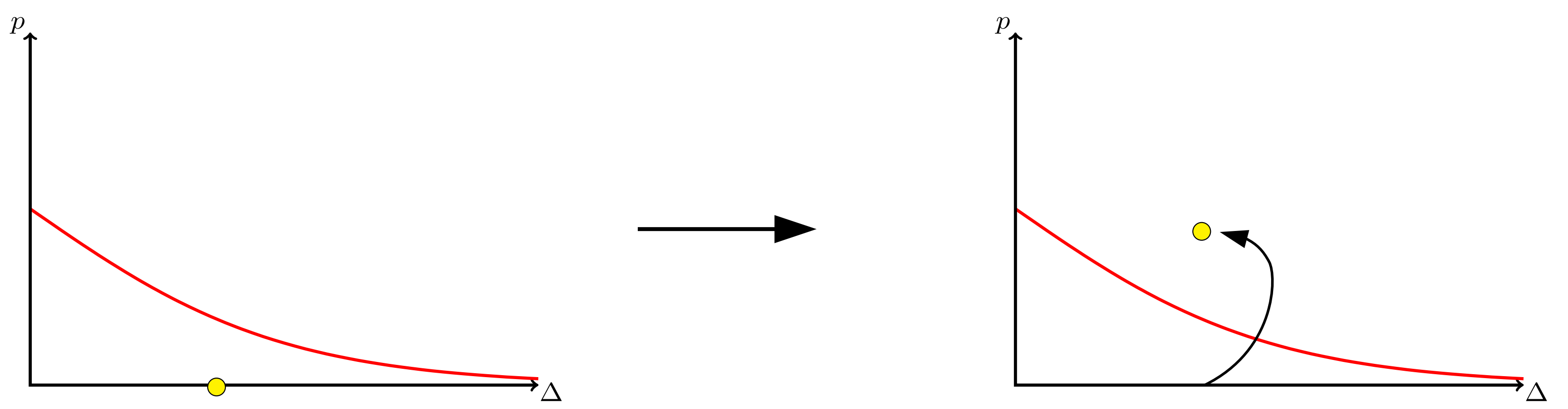
2) It preserves the thermal state:

$$\mathcal{G}_\beta[\omega_\beta(H)] = \omega_\beta(H)$$

Suffices to ensure that 2nd law cannot be violated in protocols using such thermalization maps

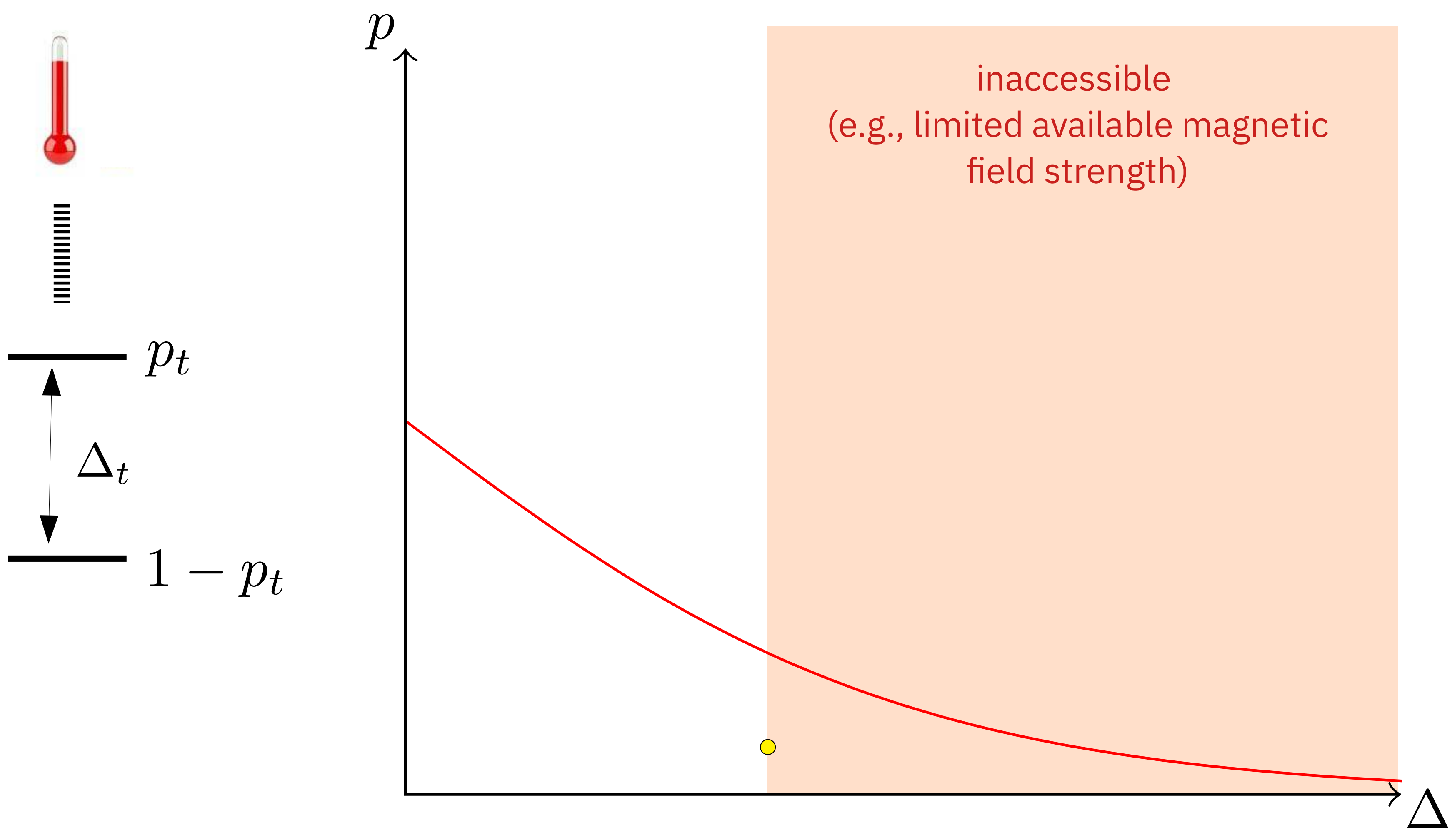
3) It induces an **approximate population inversion**:

$$\mathcal{G}_\beta[|0\rangle\langle 0|] = (1 - \mathrm{e}^{-\beta\Delta})|0\rangle\langle 0| + \mathrm{e}^{-\beta\Delta}|1\rangle\langle 1|$$



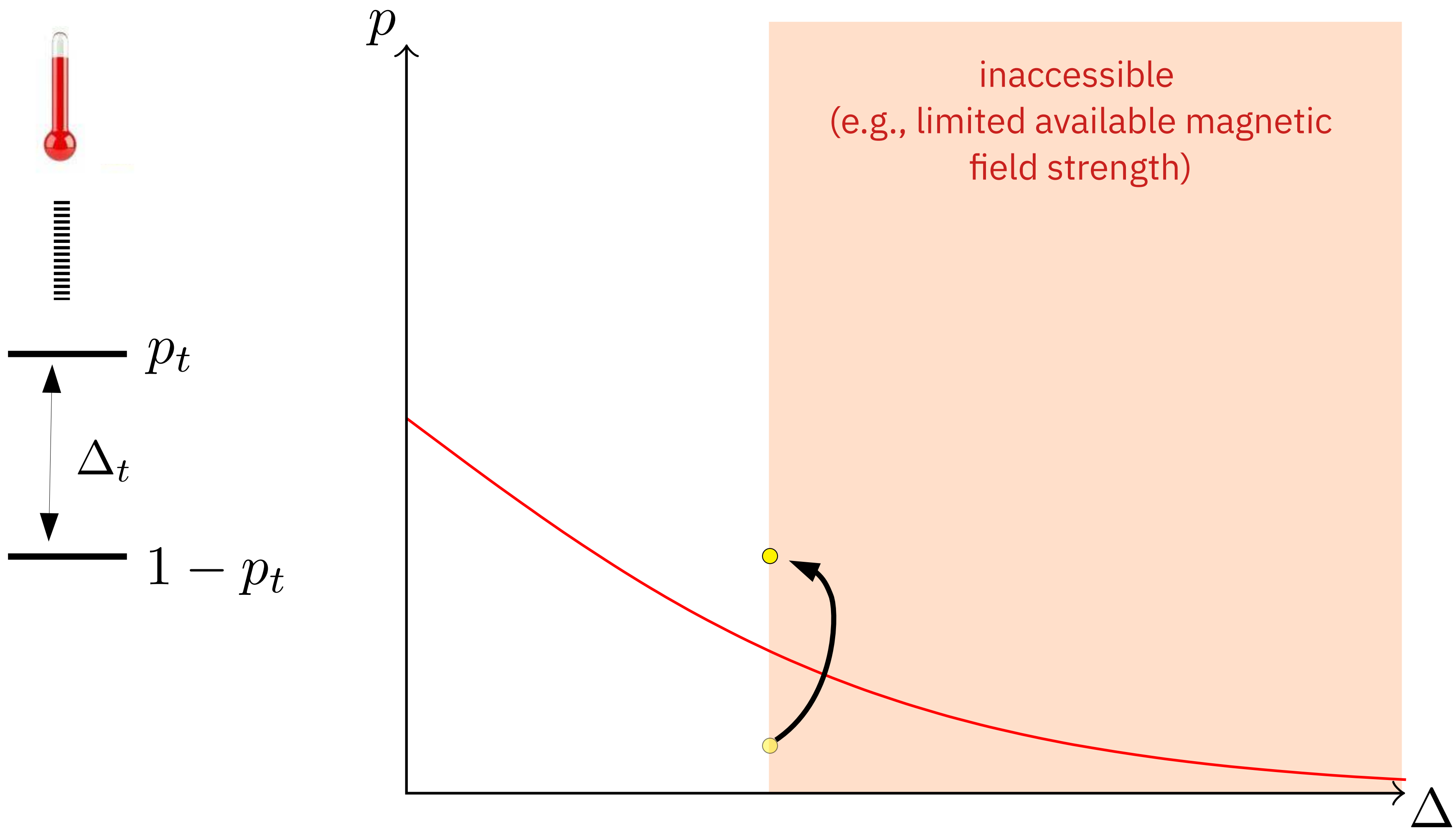
H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).
Á. M. Alhambra, M. Lostaglio, and C. Perry, Quantum 3, 188 (2019).
K. Korzekwa, M. Lostaglio, arXiv: 2005.02403.

Toy example of limited control: Work extraction using a two-level system



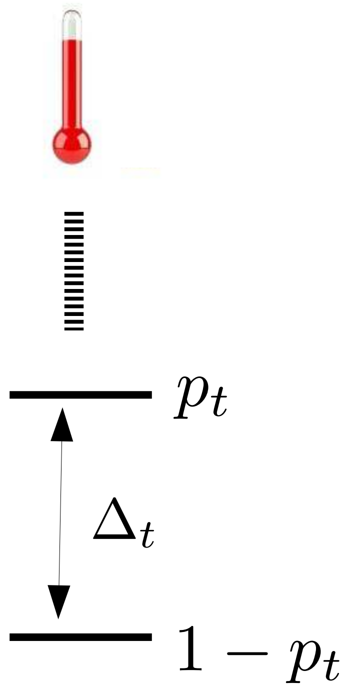
H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

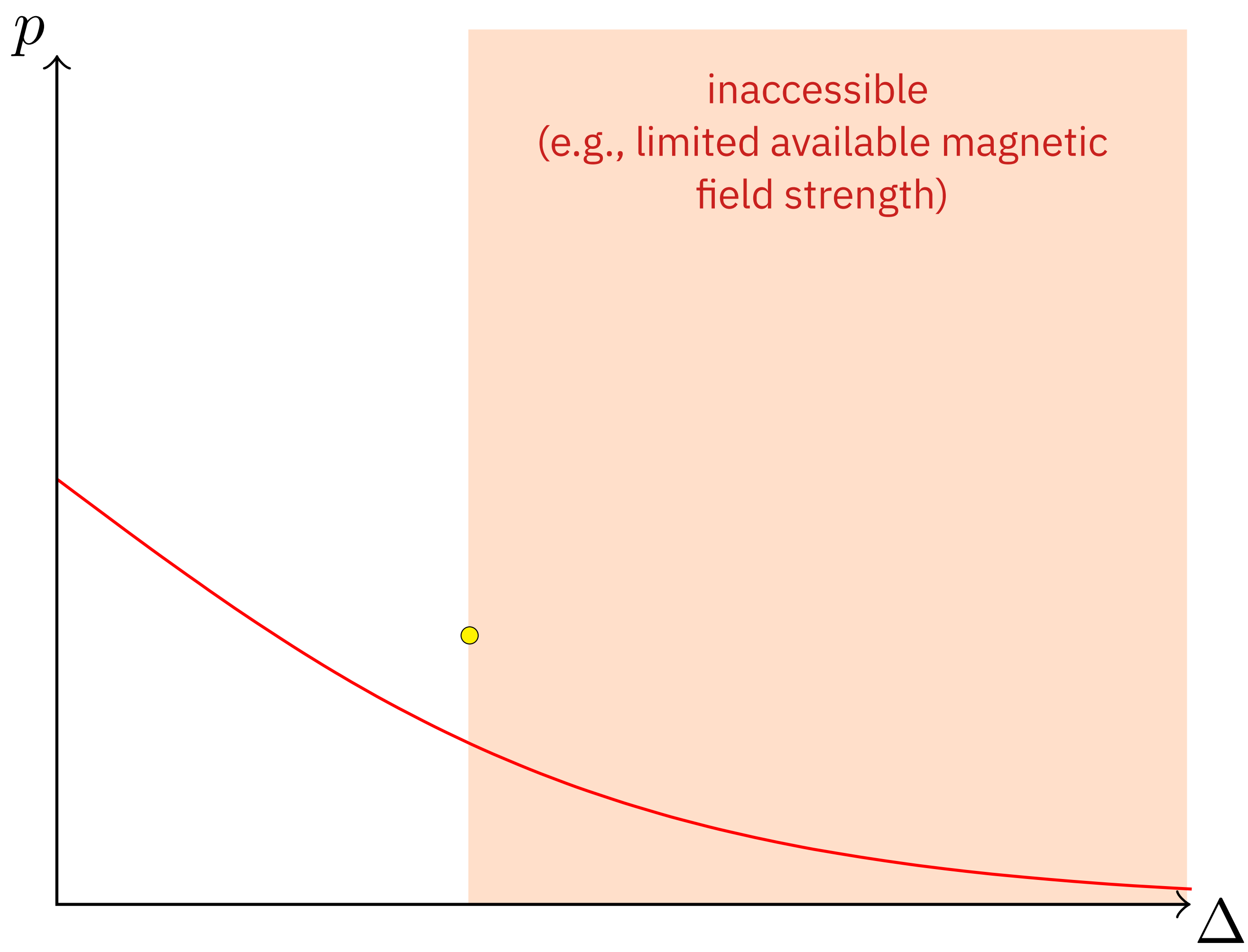
Toy example of limited control: Work extraction using a two-level system



H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

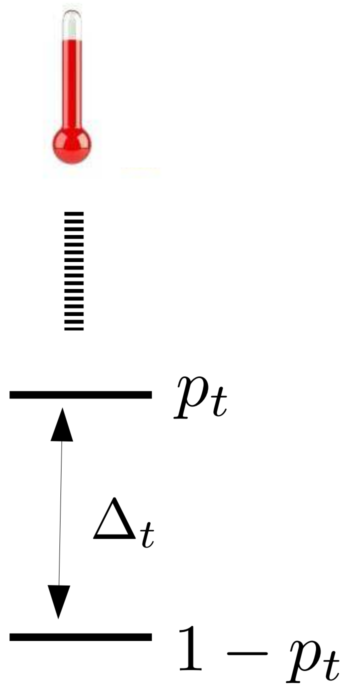
Toy example of limited control: Work extraction using a two-level system

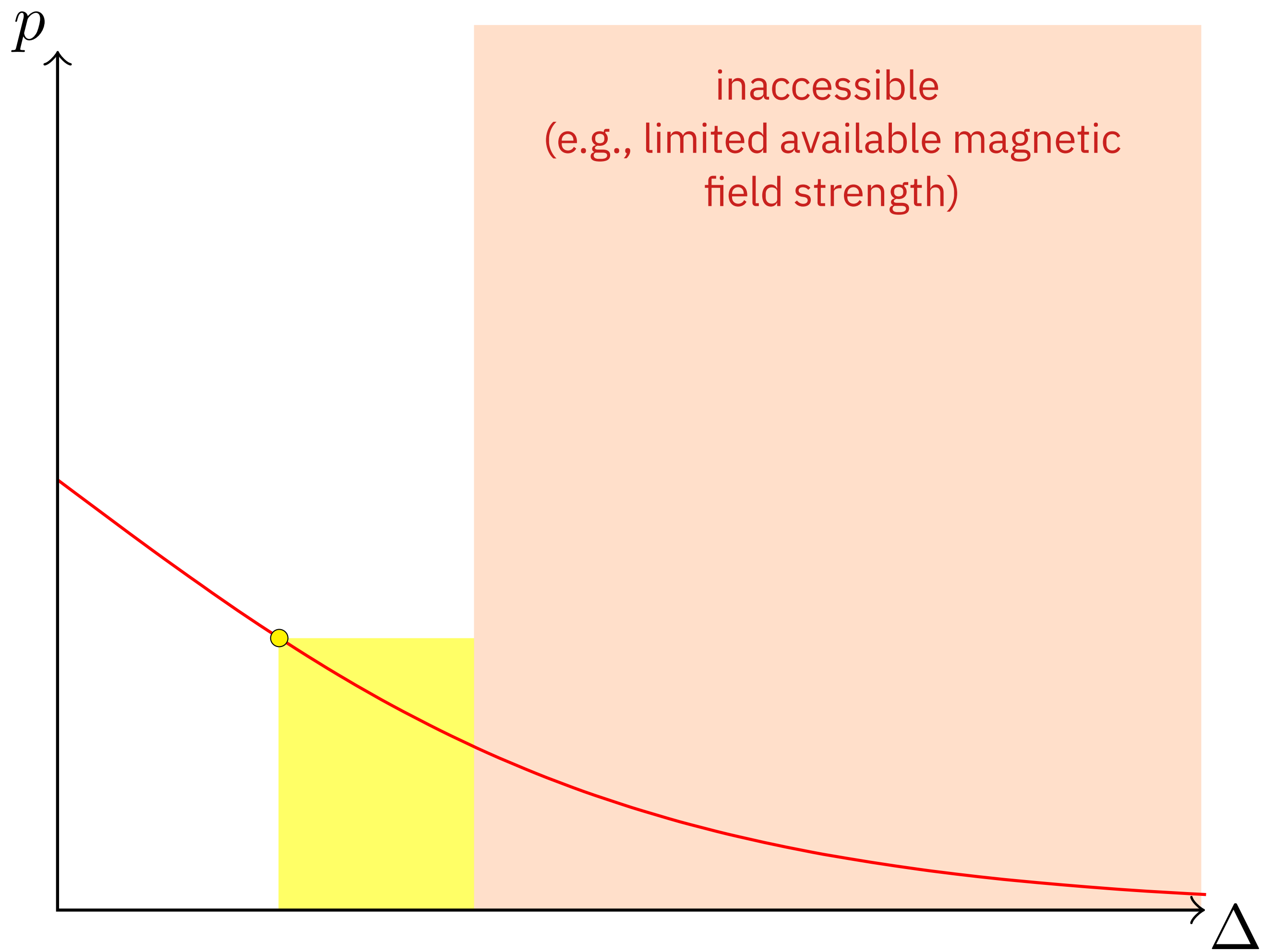




H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

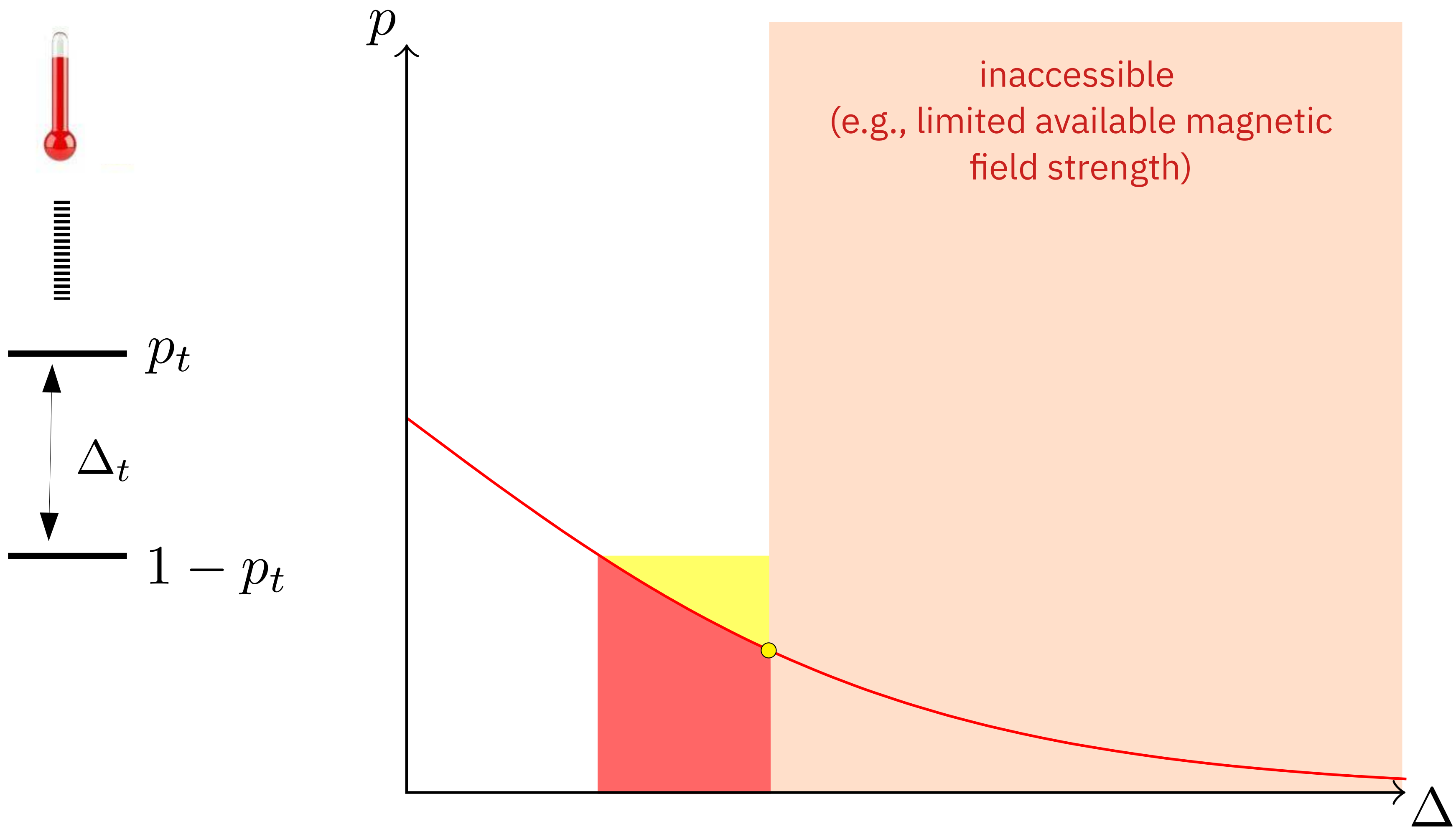
Toy example of limited control: Work extraction using a two-level system





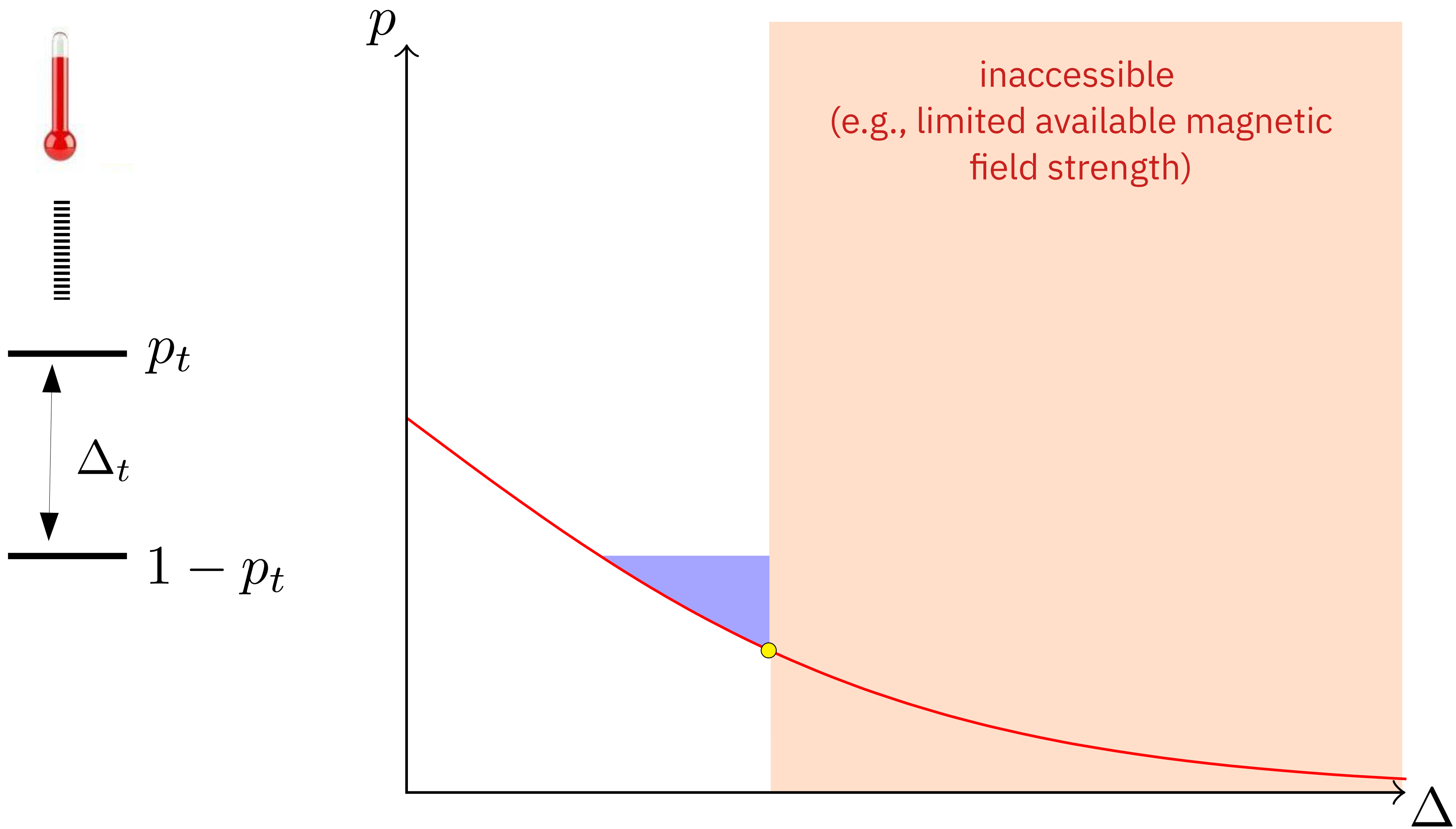
H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

Toy example of limited control: Work extraction using a two-level system



H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

Toy example of limited control: Work extraction using a two-level system

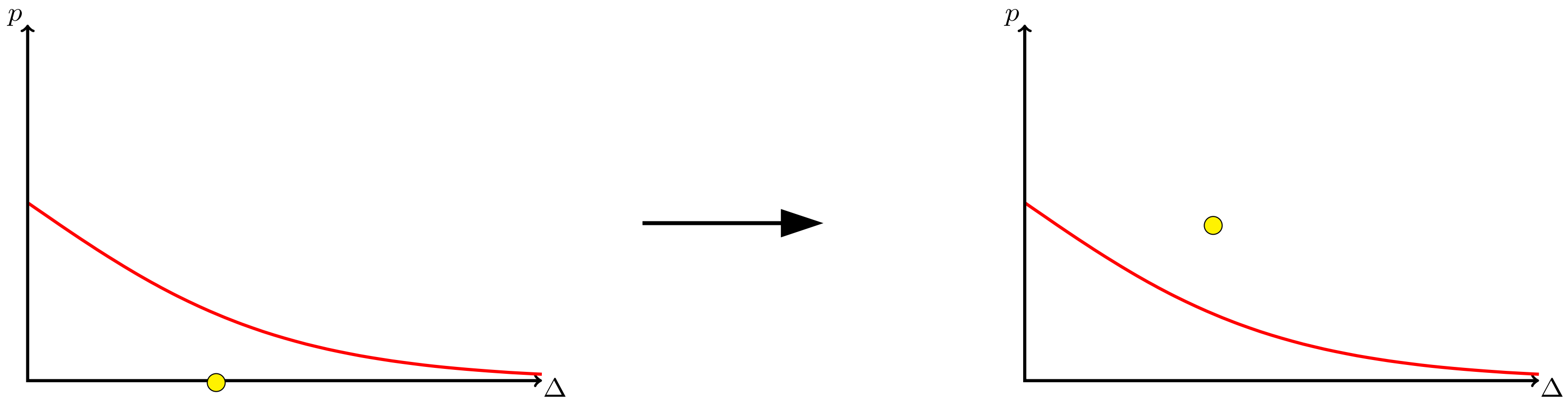


Positive work can be extracted.

Increased control over system-bath interaction can lift (to some extent) restrictions imposed by lack of control over possible Hamiltonians.

H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

Some comments on the **β-swap** $\mathcal{G}_\beta$



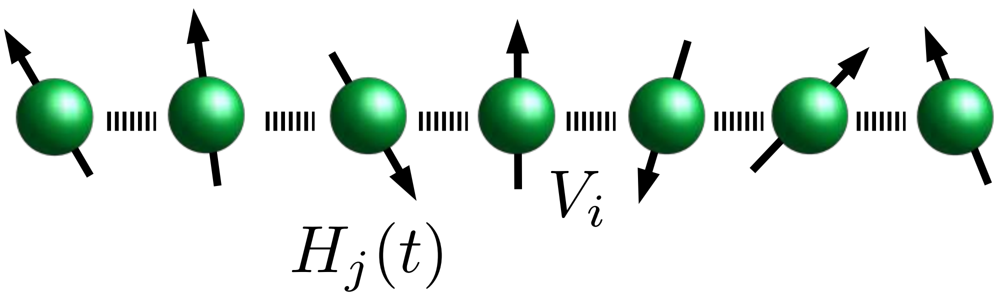
1) In **classical** setting, **cannot** be implemented by **Markovian evolution.**
(But as a stochastic matrix on system or as a permutation of energy levels of system+bath.)

2) In **quantum** setting, **can** be implemented by **Markovian evolution.**
(However, it then requires a **source of coherence** apart from the bath, since state at intermediate times is not diagonal in energy eigenbasis.)

For details, see K. Korzekwa, M. Lostaglio, arXiv: 2005.02403.

More relevant scenario:

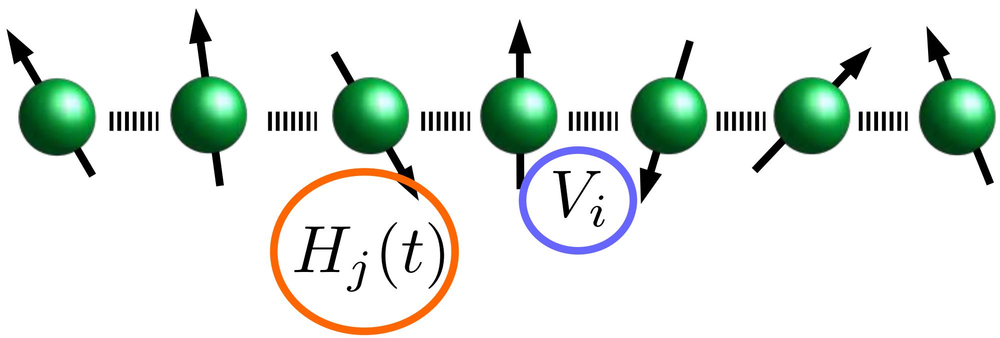
$$H_t = H_{\mathrm{ext}}(t) + H_{\mathrm{int}}$$



Local control

More relevant scenario:

$$H_t = H_{\text{ext}}(t) + H_{\text{int}}$$



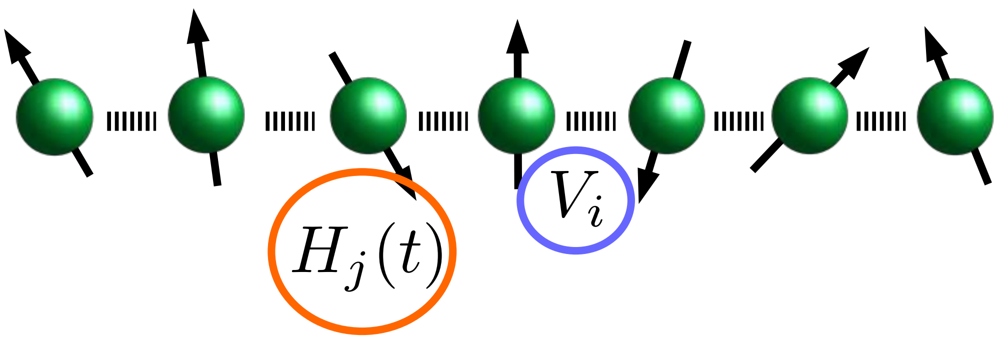
External fields: can be controlled

Internal interactions: fixed

Local control

More relevant scenario:

$$H_t = H_{\text{ext}}(t) + H_{\text{int}}$$



External fields: can be controlled

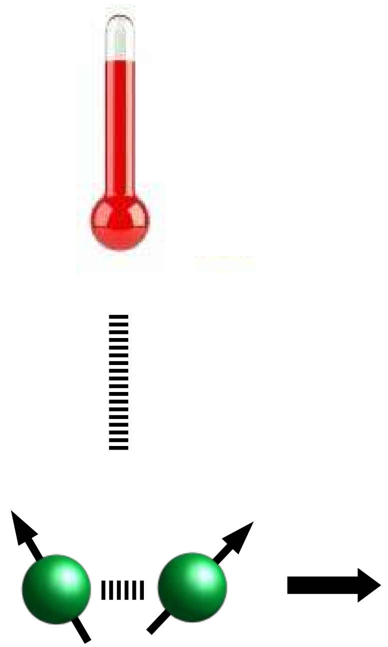
Internal interactions: fixed

General result in quantum control theory: For any "true" interaction**, arbitrary unitaries on full system** can be generated if arbitrary local fields can be applied.

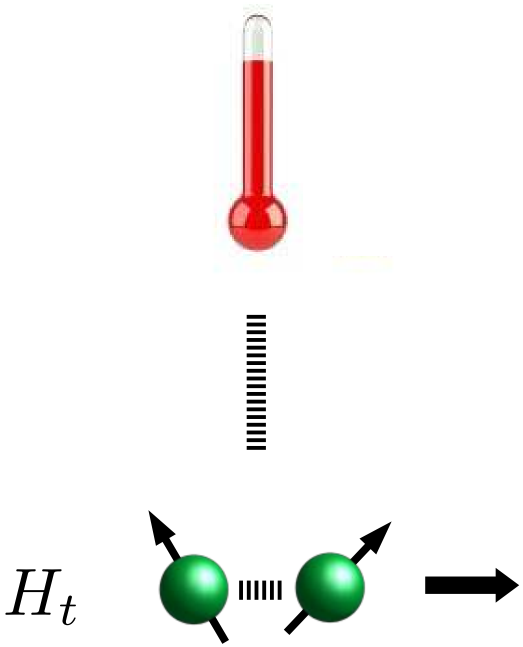
See, for example:
Lloyd, S., *Phys. Rev. Lett.* **75**, 346-349 (1995)
Janzing, D. *et al.*, *Phys. Rev.* A **65,** 022104 (2002)
Burgarth, D., *et al., Phys. Rev.* A **65,** 060305(R) (2009)

Simple example: Two Spins

H_t



Simple example: Two Spins



Theorem:

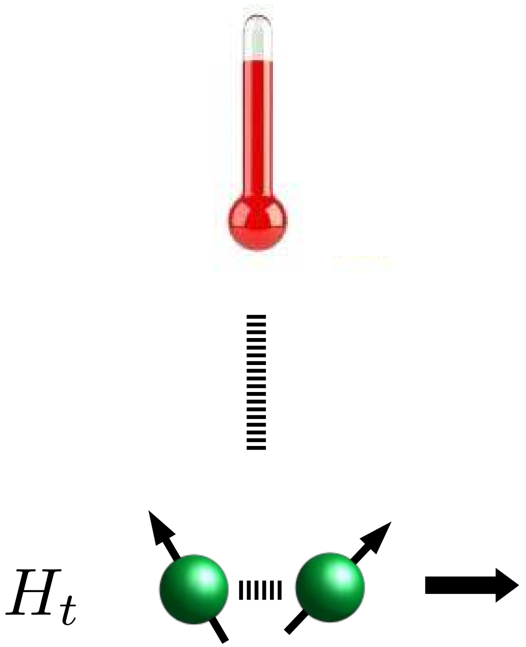
1) For the initial Hamiltonian $H_0 = \sigma_Z \otimes \sigma_Z$ and under local control, the maximally mixed initial state is **passive**.

2) There exist $\omega_\beta(H_0)$-preserving channels $\mathcal{G}_\beta$ that lift the passivity if they can be used at the beginning of the protocol.

H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

Proof somewhat complicated, but situation exactly analogous to the case of restricted field strength for single spin.

Simple example: Two Spins



Theorem:

1) For the initial Hamiltonian $H_0 = \sigma_Z \otimes \sigma_Z$ and under local control, the maximally mixed initial state is **passive**.

2) There exist $\omega_\beta(H_0)$ -preserving channels $\mathcal{G}_\beta$ that lift the passivity if they can be used at the beginning of the protocol.

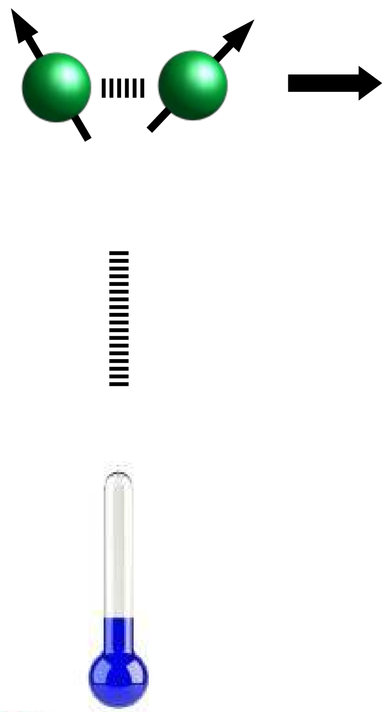
H. Wilming., R. Gallego, and J. Eisert, *Phys. Rev. E* 93.4 (2016).

Proof somewhat complicated, but situation exactly analogous to the case of restricted field strength for single spin.

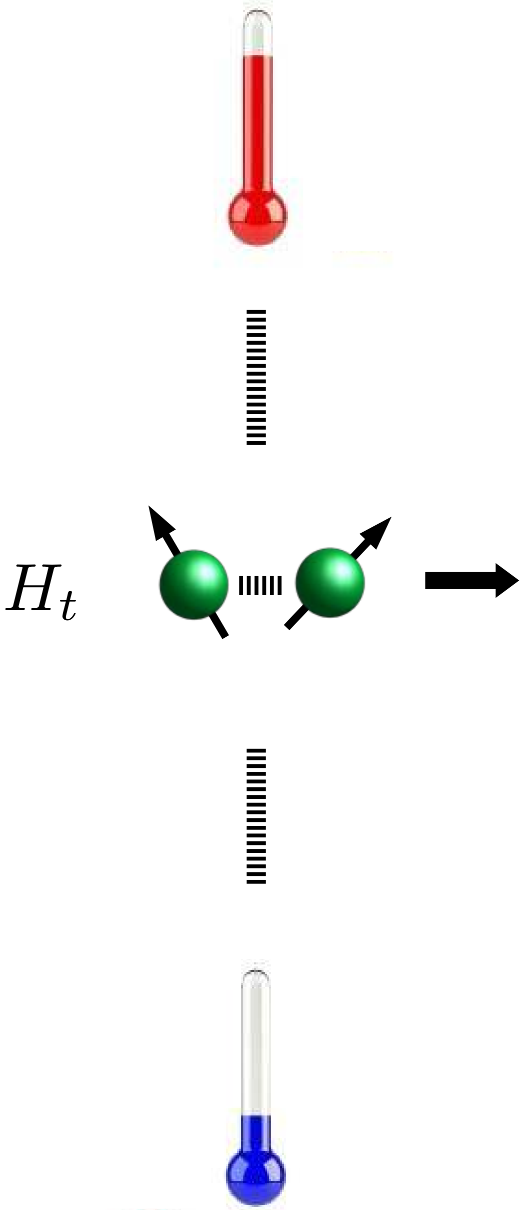
What effects does restriction to local control have in **many-body systems?**

The Second Law under control restrictions

H_t



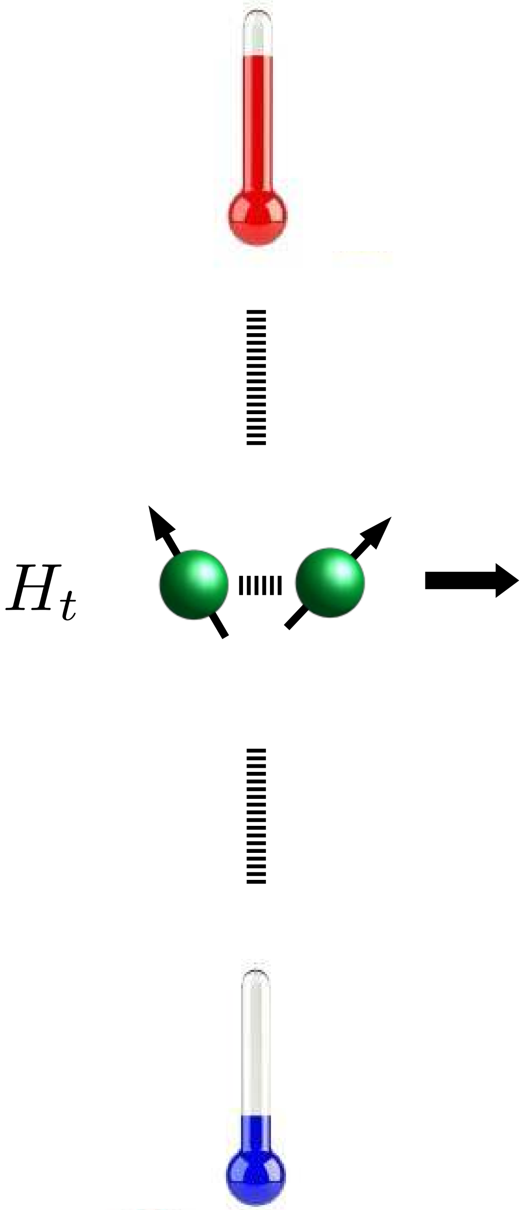
J. Lekscha, H. Wilming, J. Eisert, and R. Gallego, *Phys. Rev. E 97 (2), 022142* (2018).



Theorem (General bound on efficiency):
The maximum efficiency of an engine operated under restricted control is given by

$$\eta = 1 - \frac{T_c}{T_h}\left[\min_{\substack{U_h, U_c,\\ H^{(1)},\ldots,H^{(4)}}} \frac{\Delta S + \mathrm{Dissip.}^c}{\Delta S - \mathrm{Dissip.}^h}\right]$$

J. Lekscha, H. Wilming, J. Eisert, and R. Gallego, *Phys. Rev. E 97 (2), 022142* (2018).



Theorem (General bound on efficiency):
The maximum efficiency of an engine operated under restricted control is given by

$$\eta = 1 - \frac{T_c}{T_h}\left[\min_{\substack{U_h, U_c,\\ H^{(1)},\ldots,H^{(4)}}} \frac{\Delta S + \mathrm{Dissip.}^c}{\Delta S - \mathrm{Dissip.}^h}\right]$$

$$\mathrm{Dissip.}^c = S(U_c \omega_{\beta_h}(H^{(2)}) U_c^\dagger \| \omega_{\beta_c}(H^{(3)}))$$

$$\mathrm{Dissip.}^h = S(U_h \omega_{\beta_c}(H^{(4)}) U_h^\dagger \| \omega_{\beta_h}(H^{(1)}))$$

$$\Delta S = S(\omega_{\beta_h}(H^{(2)})) - S(\omega_{\beta_c}(H^{(4)}))$$

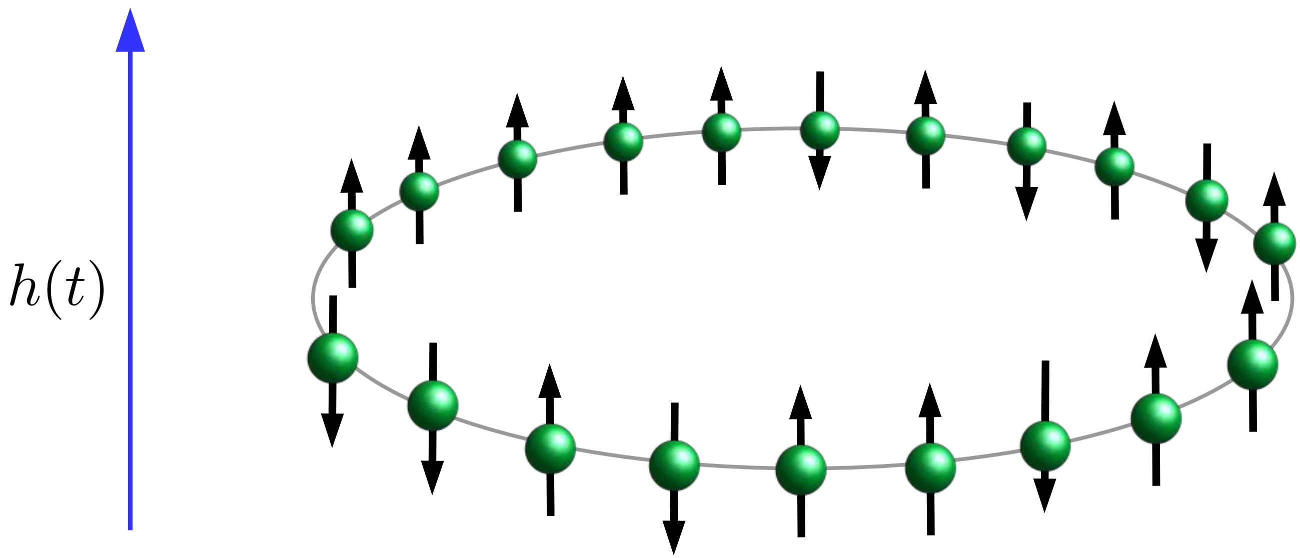
$H^{(i)}$: Hamiltonians of the form $H_t = H_{\mathrm{ext}}(t) + H_{\mathrm{int}}$

$U_{c/h}$: Arbitrary unitaries that can be obtained by evolving under the possible Hamiltonians

J. Lekscha, H. Wilming, J. Eisert, and R. Gallego, *Phys. Rev. E 97 (2), 022142* (2018).

Case study: Classical 1-D Ising model with periodic boundary conditions

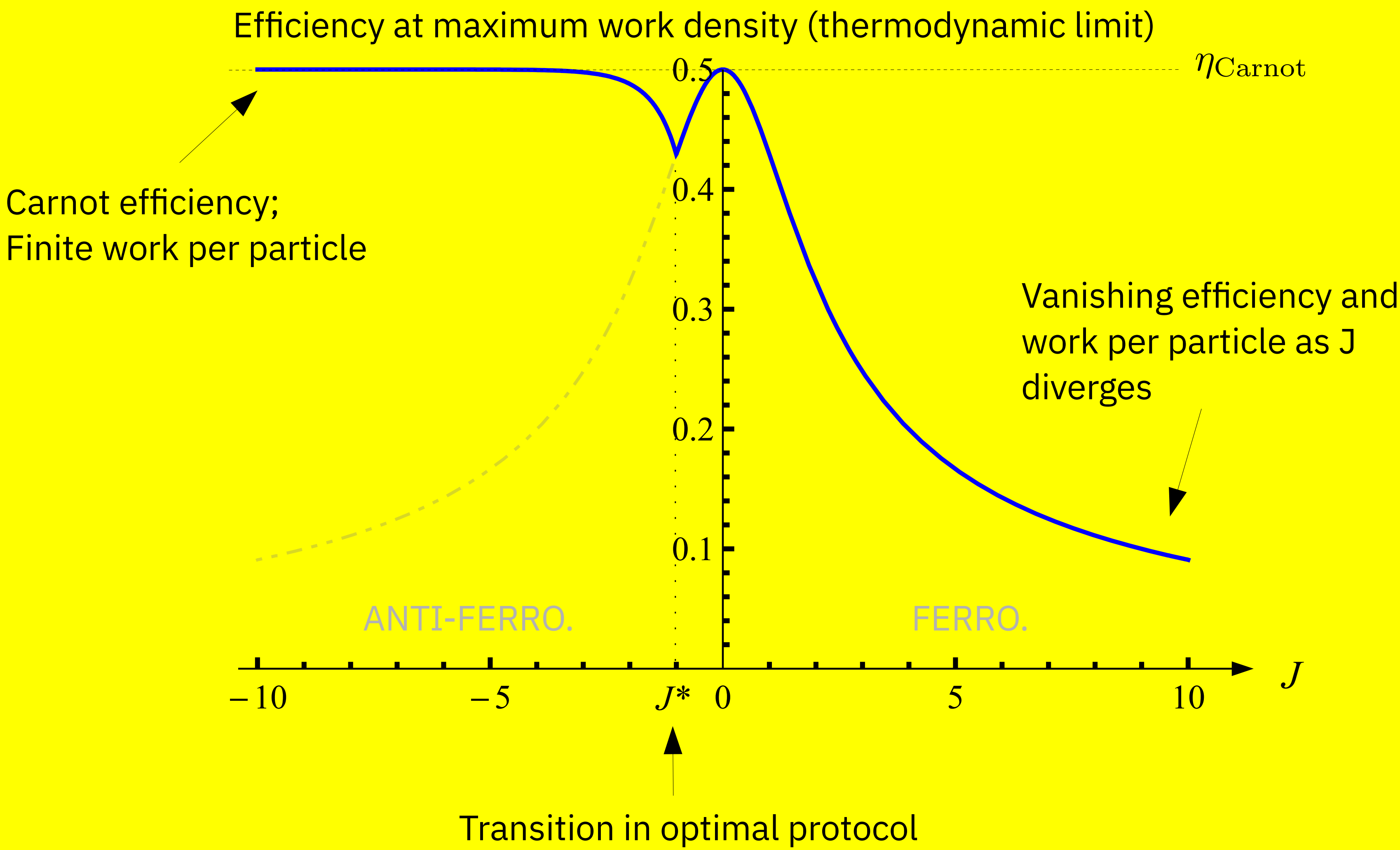
$$H_N = h(t) \sum_{j=1}^{N} \sigma_z^{(j)} - J \sum_{j=1}^{N} \sigma_z^{(j)} \sigma_z^{(j+1)}$$



How does the performance depend on the interaction strength?
(Energy scales introduced by temperatures of heat baths)

Case study: Classical 1-D Ising model with periodic boundary conditions

$$H_N = h(t) \sum_{j=1}^{N} \sigma_z^{(j)} - J \sum_{j=1}^{N} \sigma_z^{(j)} \sigma_z^{(j+1)}$$



J. Lekscha, H. Wilming, J. Eisert, and R. Gallego, *Phys. Rev. E 97 (2), 022142* (2018).

Summary

- Limitations on the implementable Hamiltonians can lead to situations where **no work can be extracted from non-equilibrium condition** even if reachable Hamiltonians still allow to implement **arbitrary unitary processes on working system**.

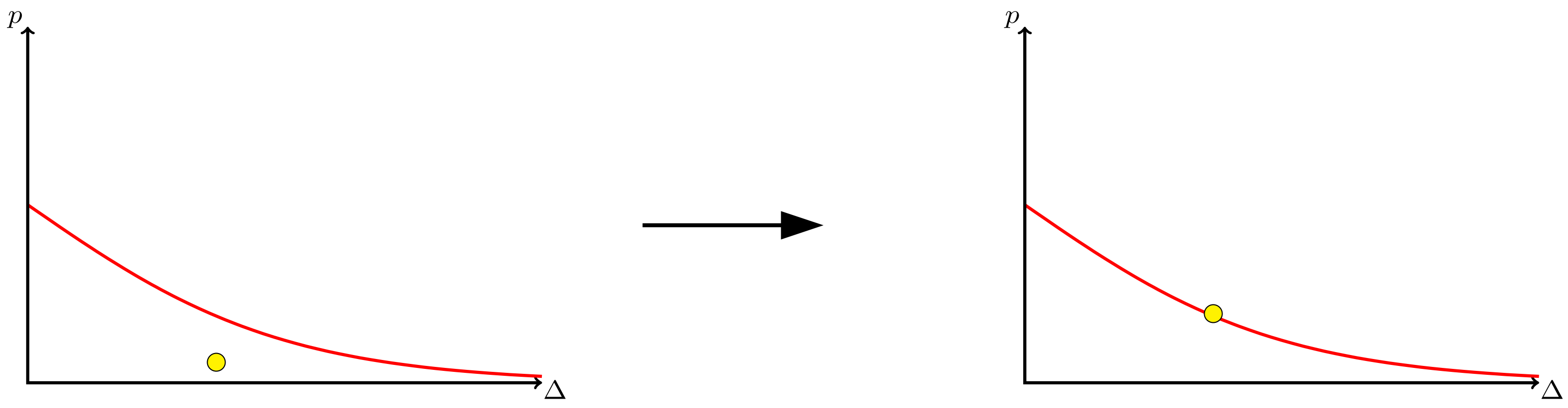
- This limitation may be overcome if more fine-grained control over the heat baths is available. Can be possible using **small heat baths**, consisting of a few particles.

- (Not shown today) Related effects can arise in strong coupling settings and with heat baths that thermalize to Generalized Gibbs Ensembles (GGEs).

Some open questions/problems:

- For small systems, it could be more important to optimize fluctuations instead of mean values. How do control restrictions enter fluctuation theorems?

- More systematic study of how locality constrains thermodynamics.



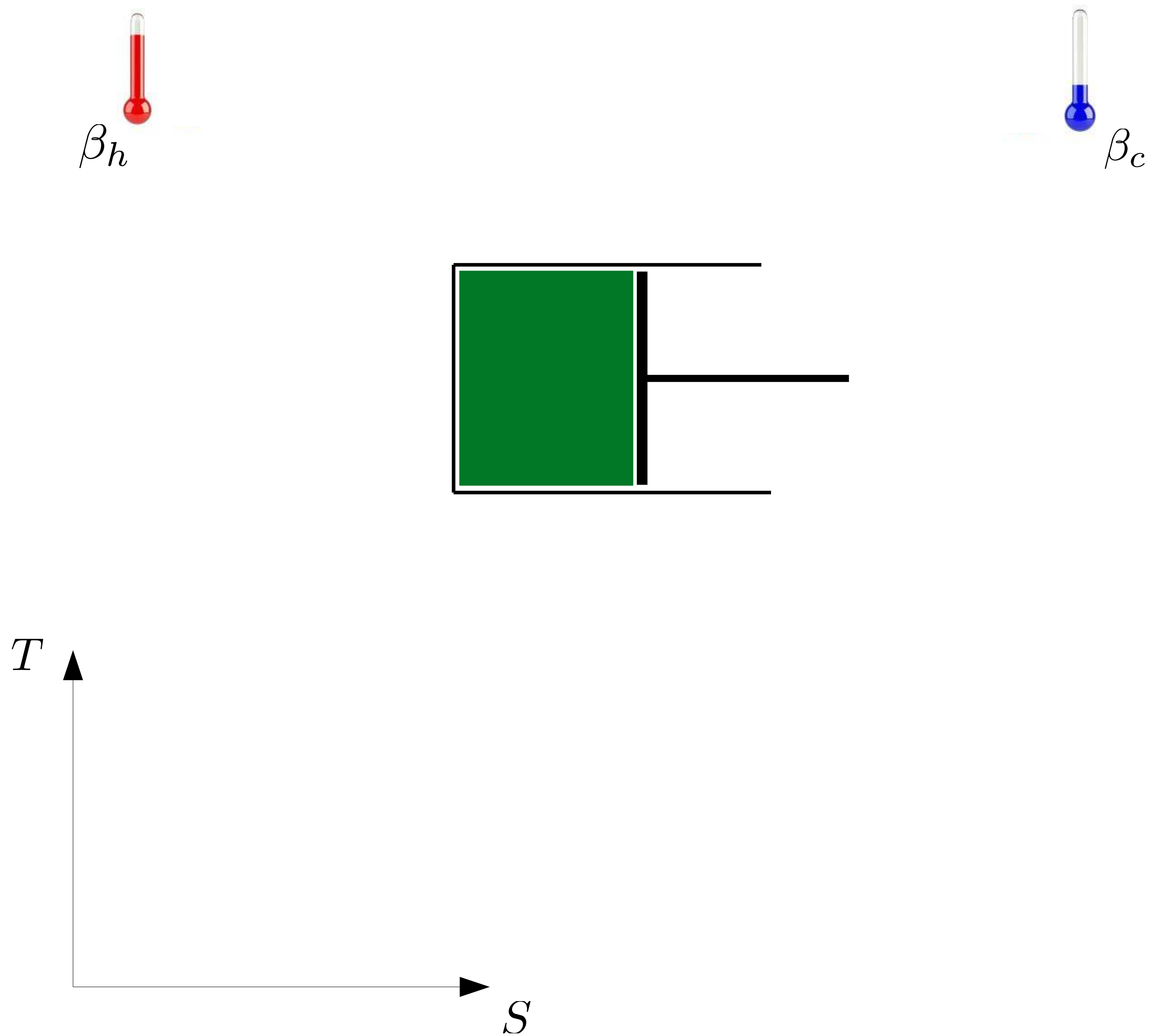
Thanks a lot for the nice workshop and your attention!

References for more detail:
H. Wilming., R. Gallego, and J. Eisert. *Phys. Rev. E* 93.4 (2016).
J. Lekscha, H. Wilming, J. Eisert, and R. Gallego. *Phys. Rev. E 97 (2), 022142* (2018).

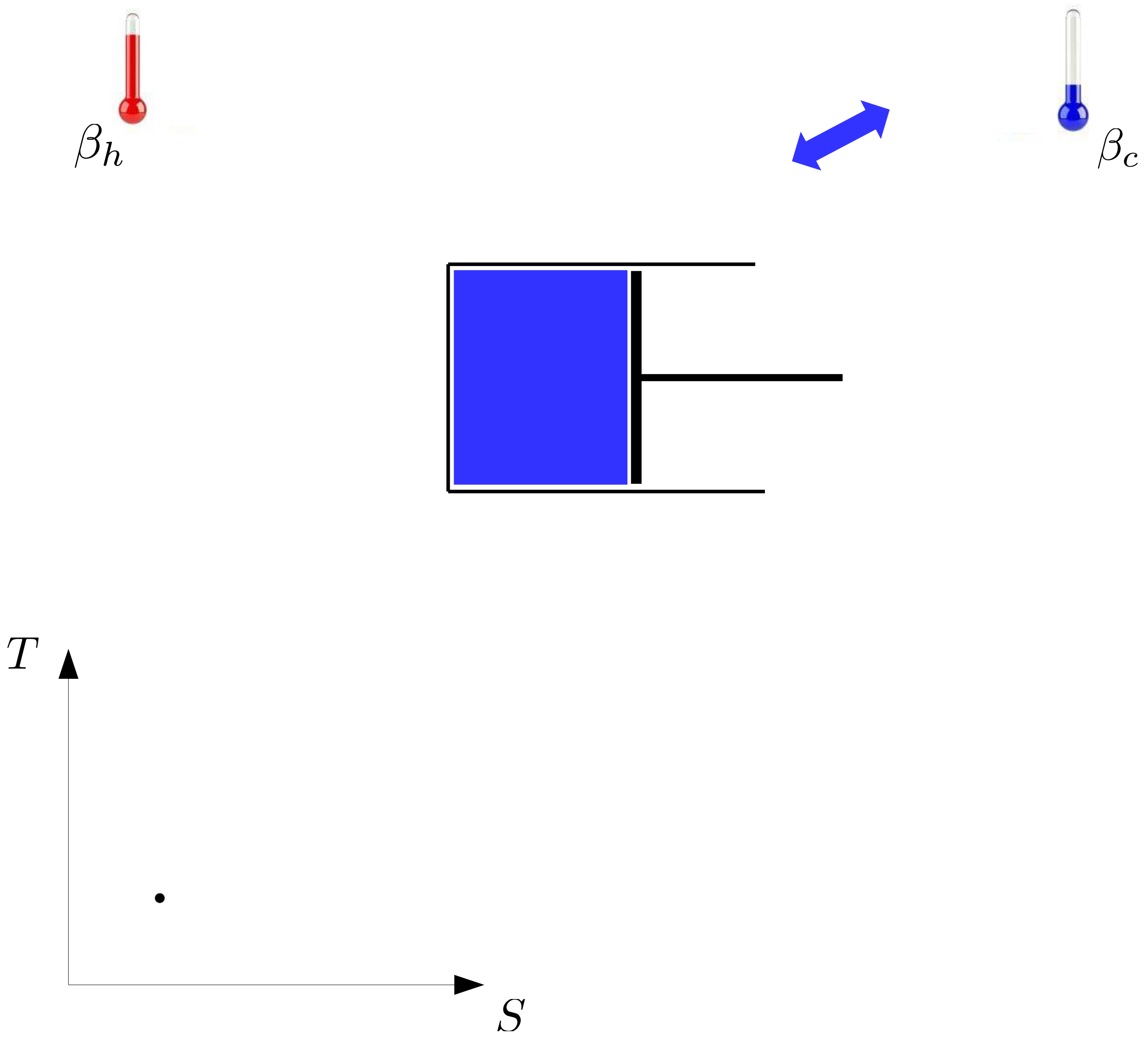
Generalized Gibbs ensembles and strong-coupling:
M. Perarnau-Llobet, A. Riera, R. Gallego, H. Wilming, and J. Eisert. *New J. Phys. 18.12* (2016)
M. Perarnau-Llobet, H. Wilming, A. Riera, R. Gallego, and J. Eisert. *Phys. Rev. Lett. 120, 120602* (2018)

Appendix

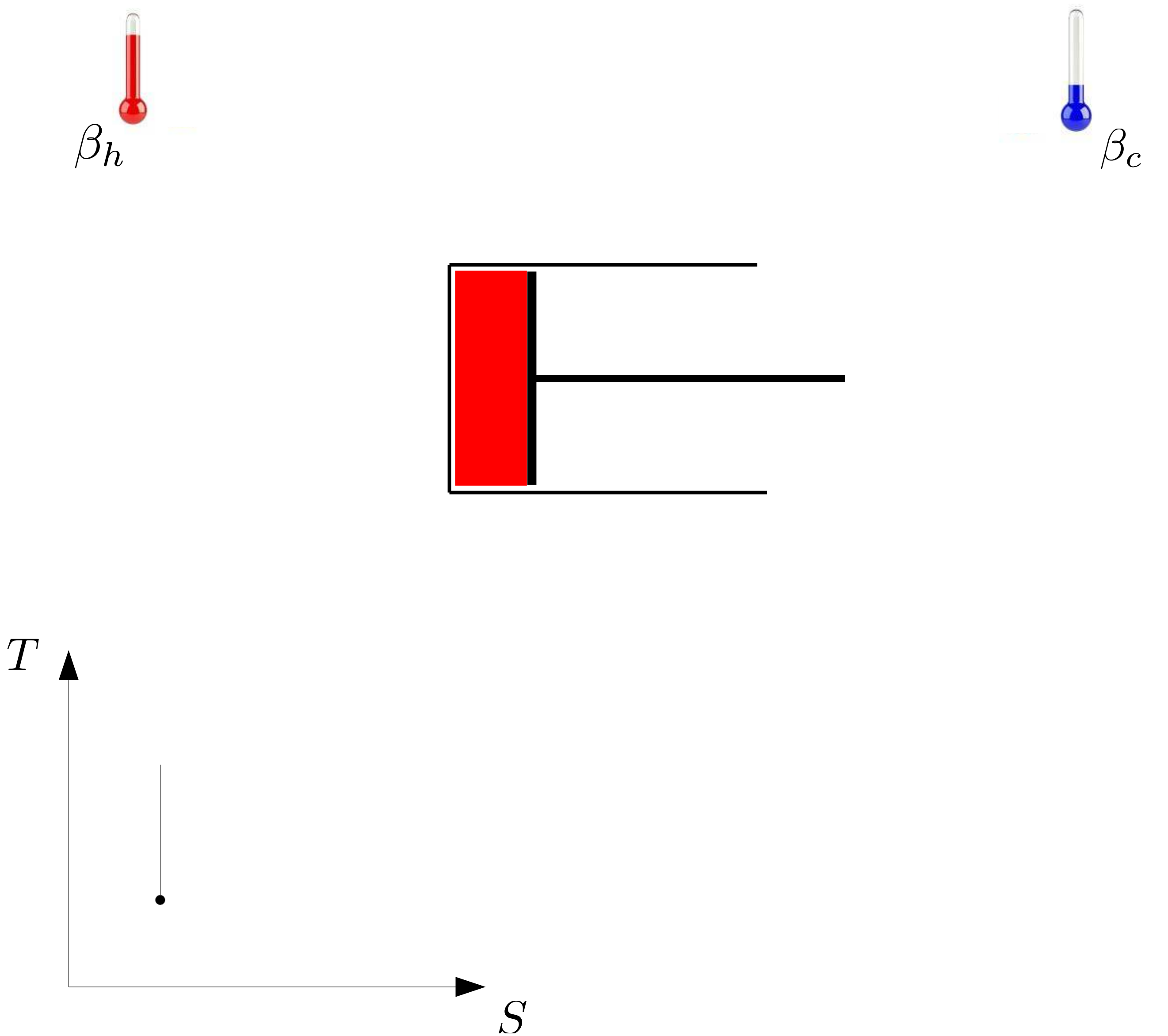
Limitations on the set of Hamiltonians. A toy model of the toy models.



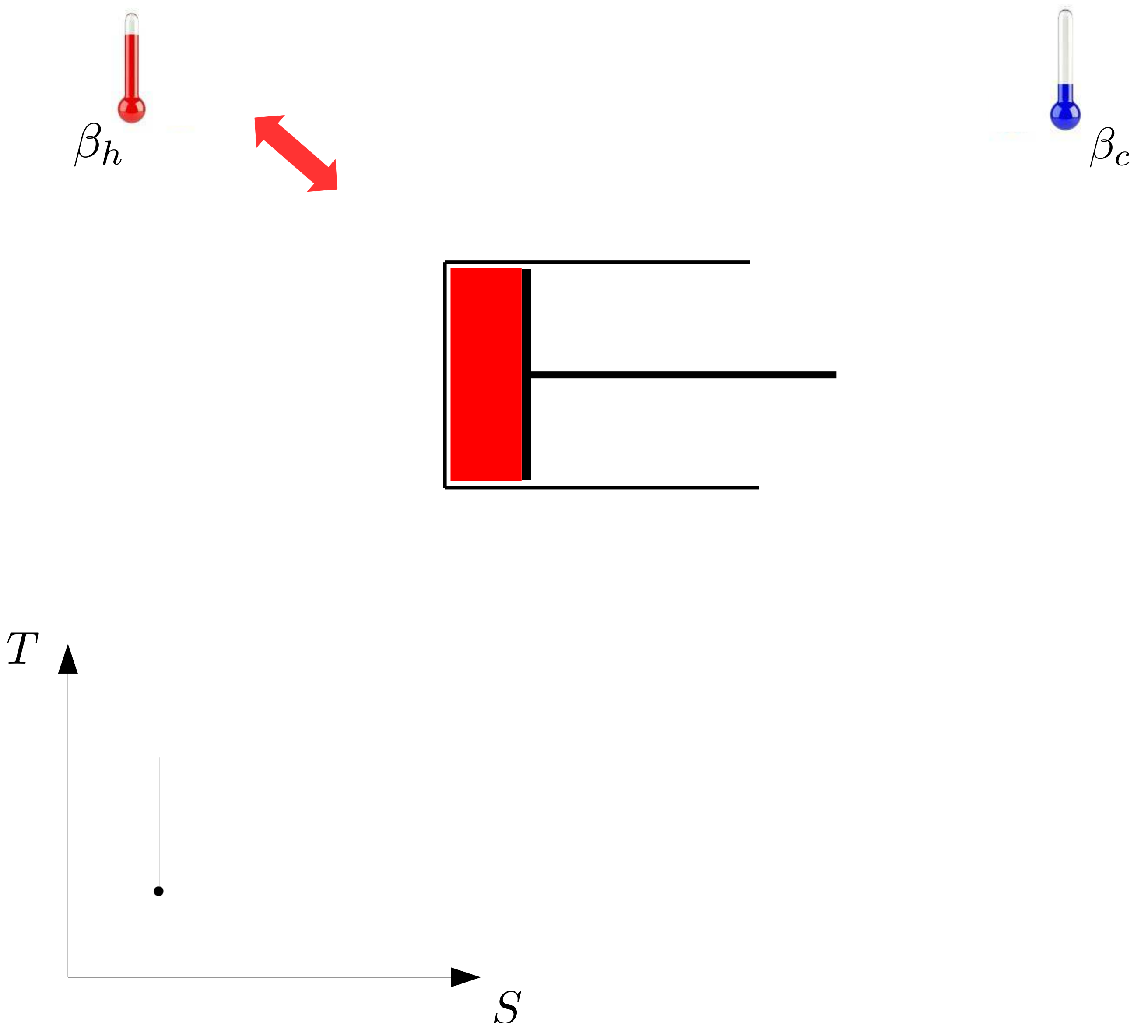
Limitations on the set of Hamiltonians. A toy model of the toy models.



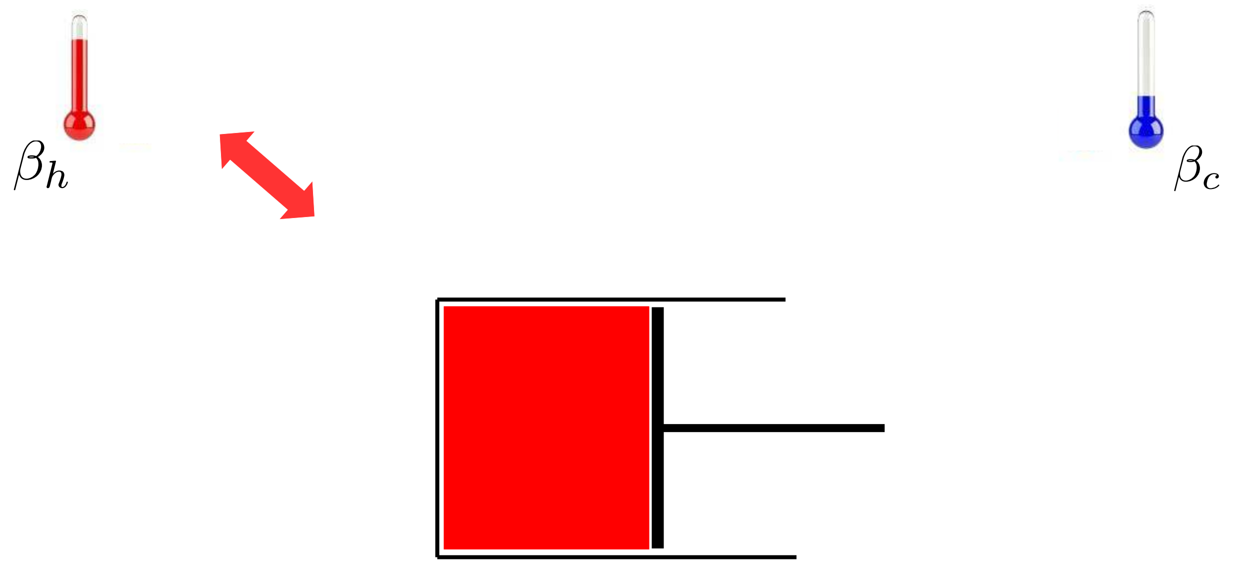
Limitations on the set of Hamiltonians. A toy model of the toy models.

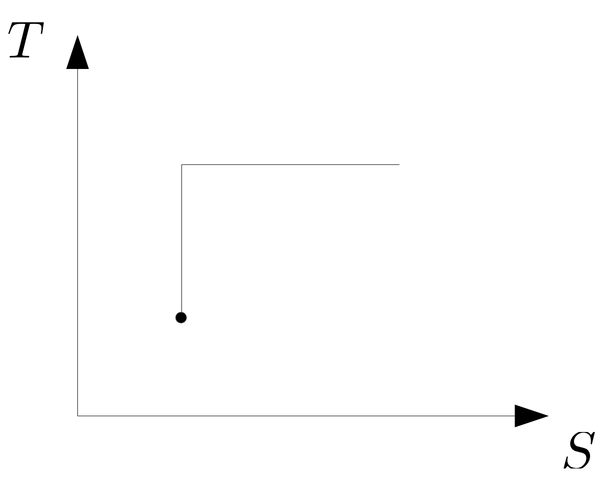


Limitations on the set of Hamiltonians. A toy model of the toy models.

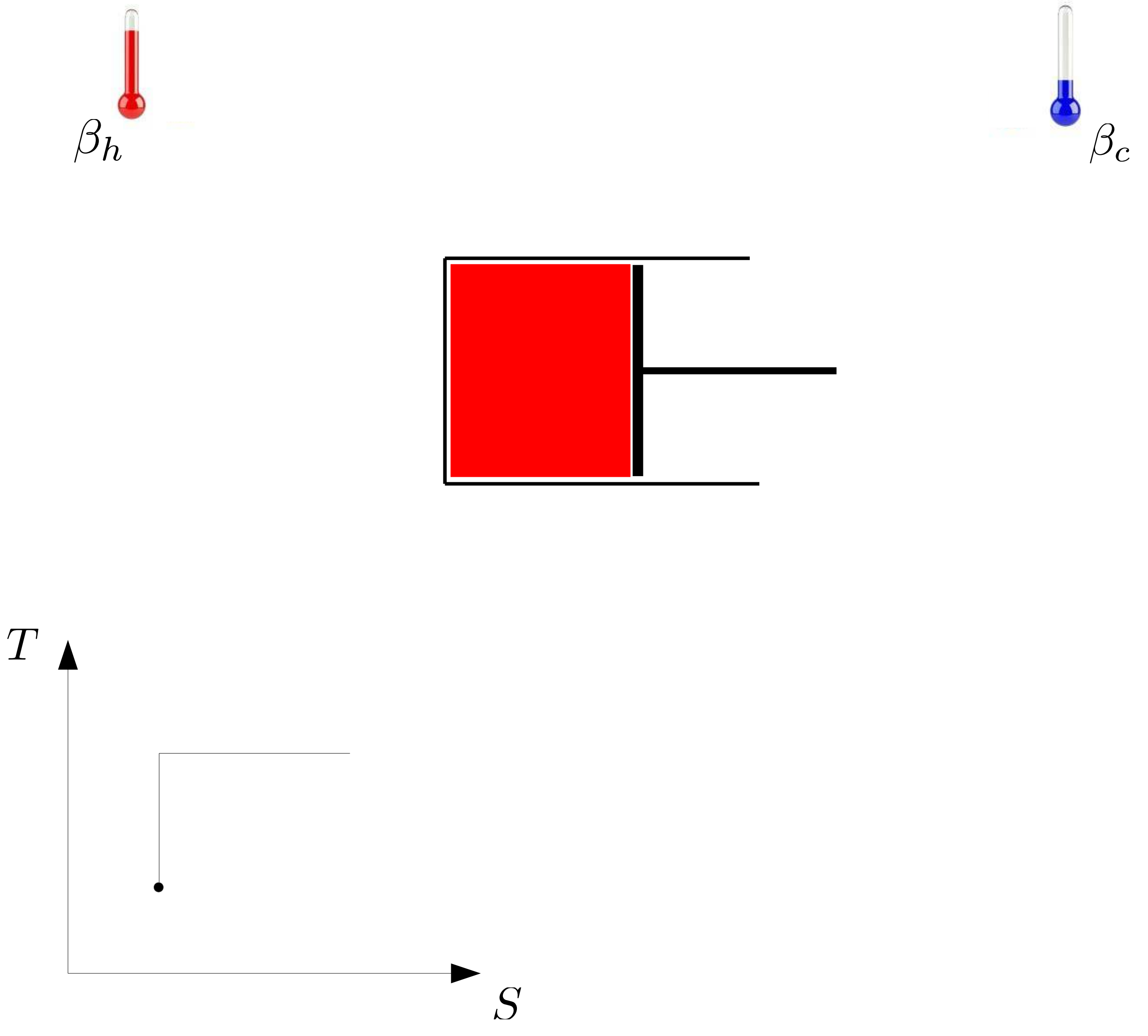


Limitations on the set of Hamiltonians. A toy model of the toy models.

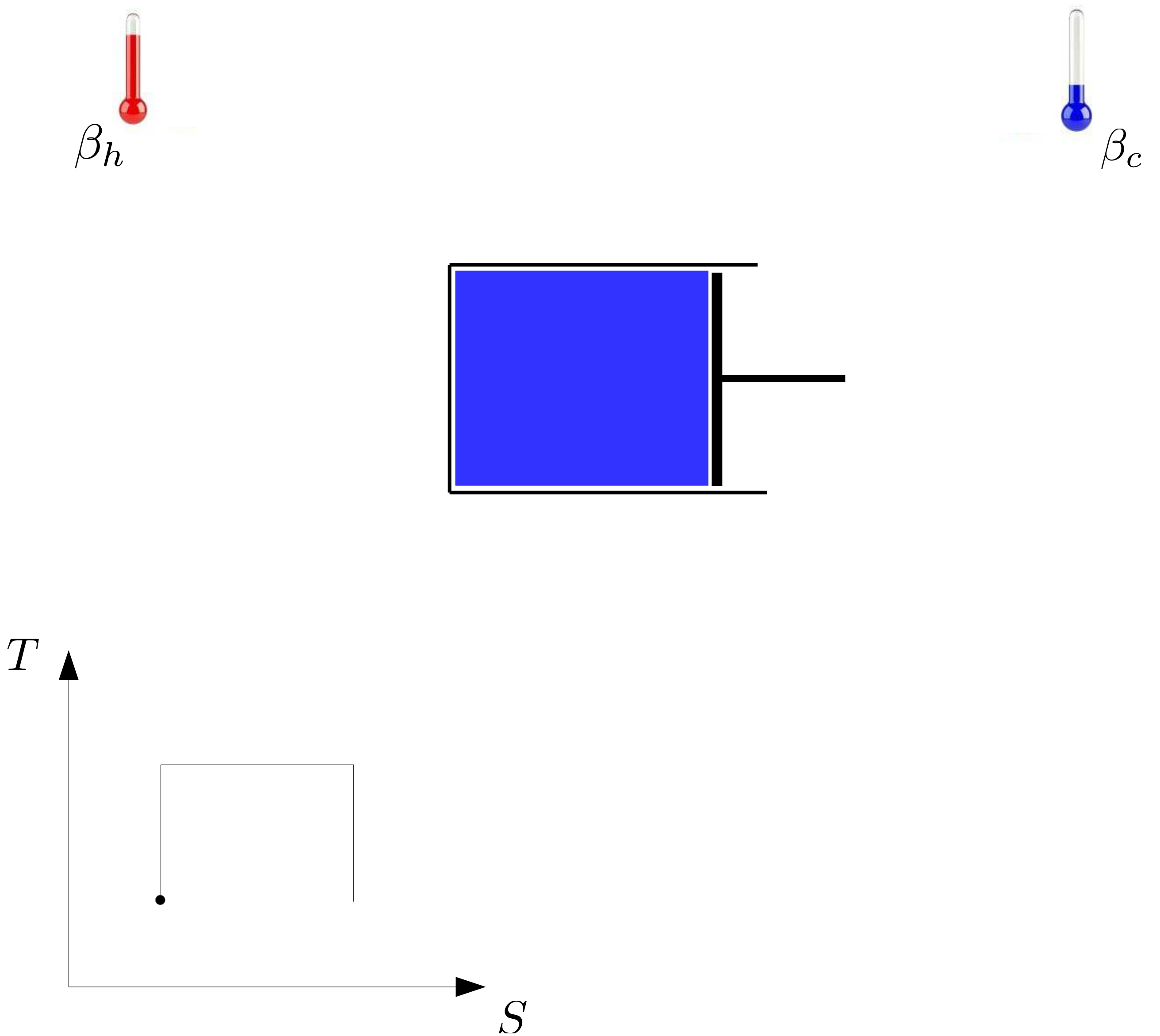




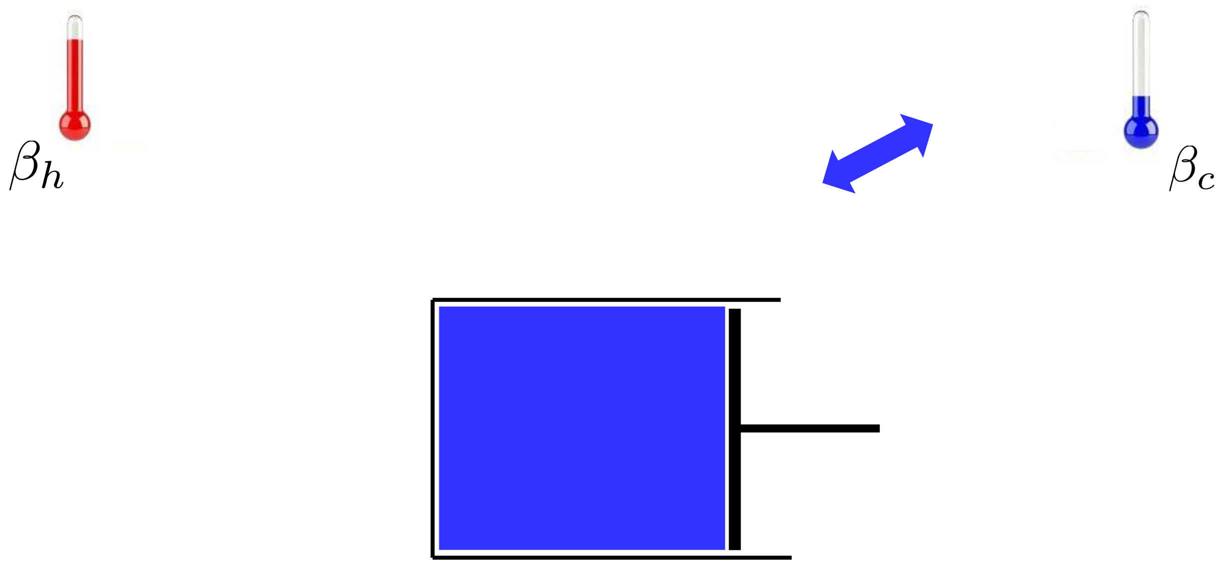
Limitations on the set of Hamiltonians. A toy model of the toy models.

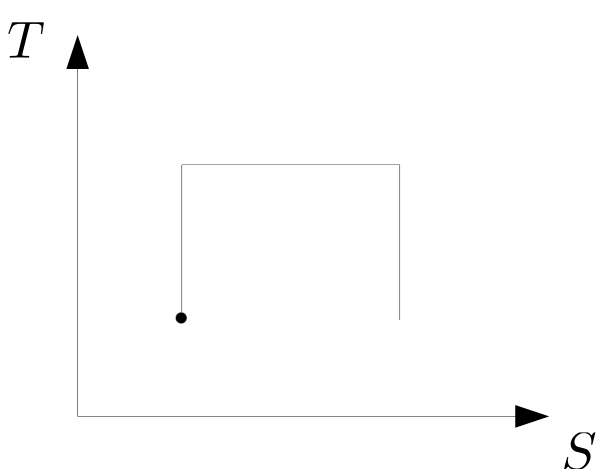


Limitations on the set of Hamiltonians. A toy model of the toy models.

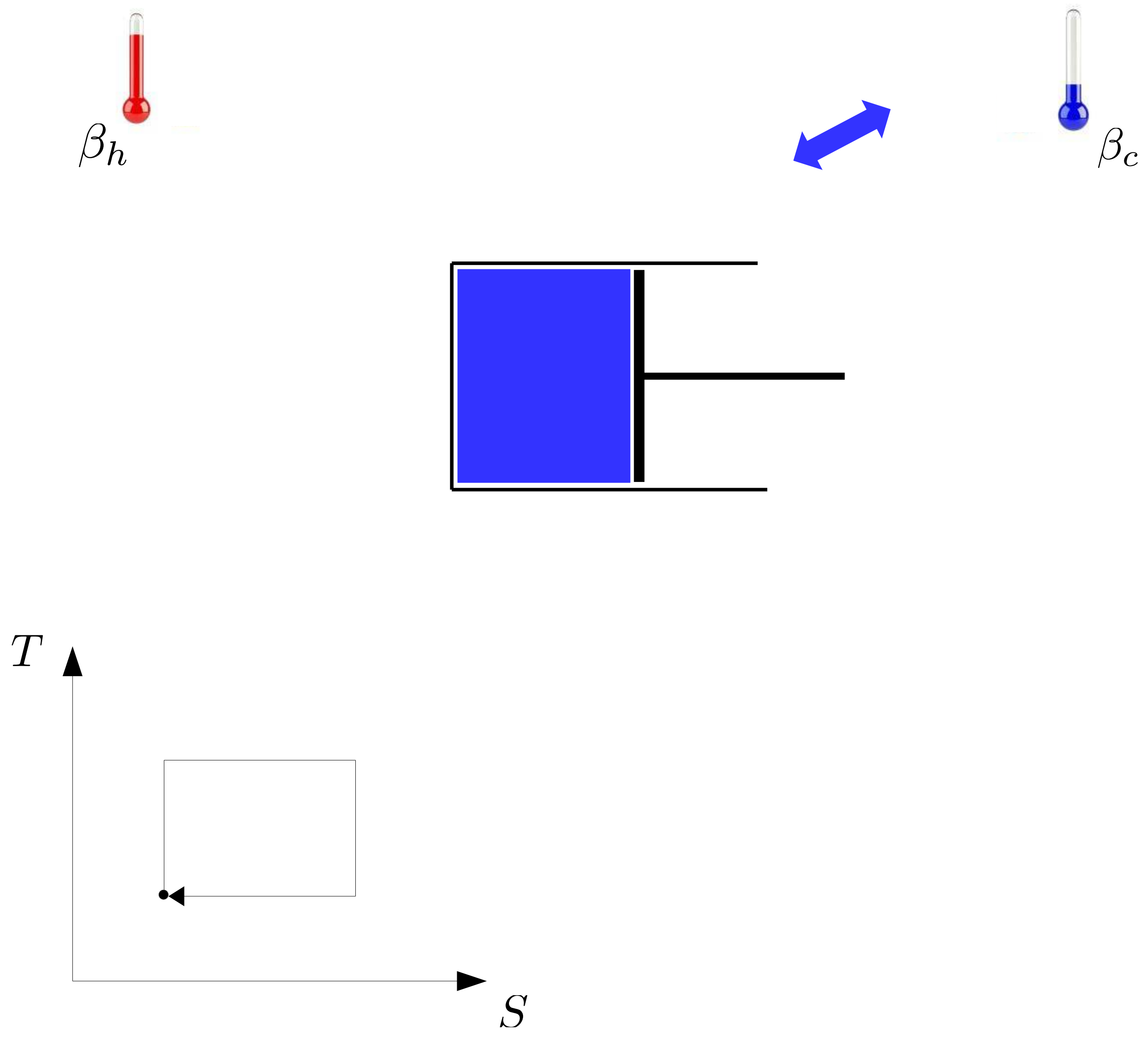


Limitations on the set of Hamiltonians. A toy model of the toy models.

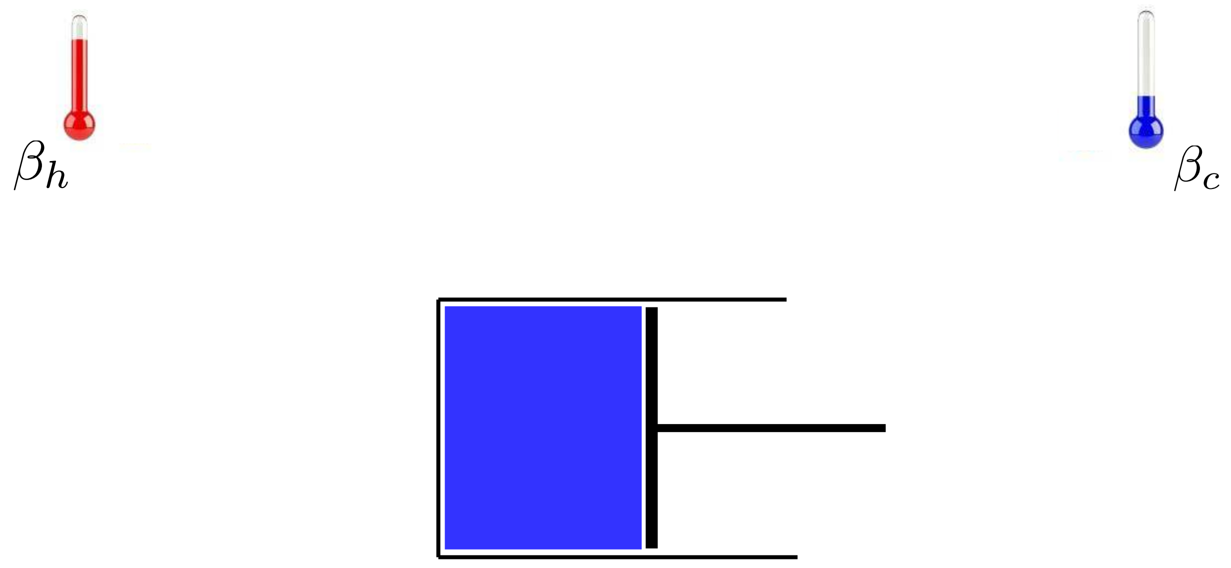


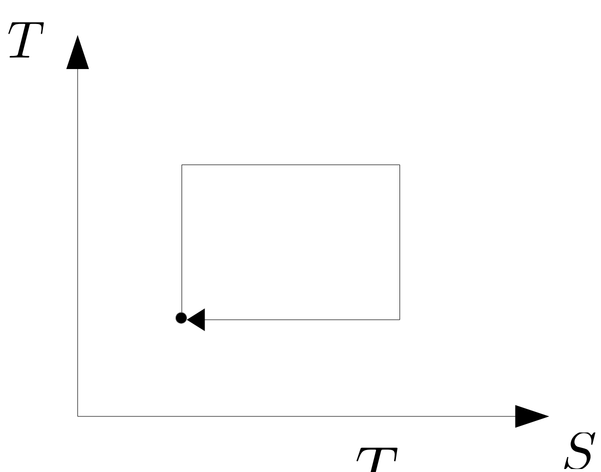


Limitations on the set of Hamiltonians. A toy model of the toy models.



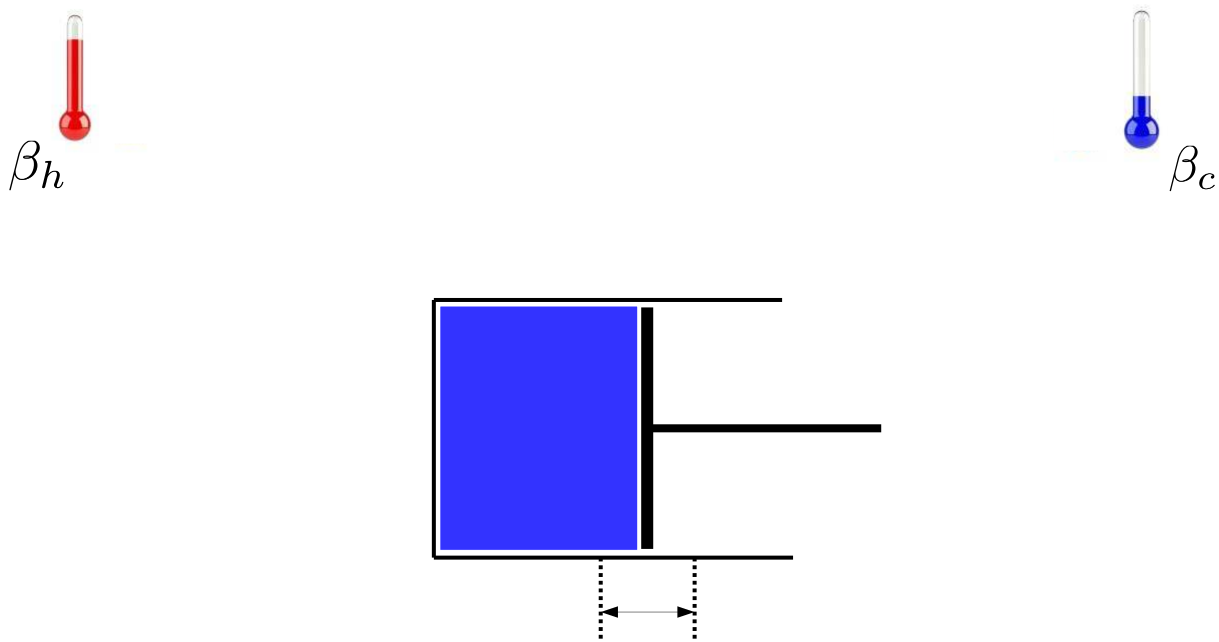
Limitations on the set of Hamiltonians. A toy model of the toy models.

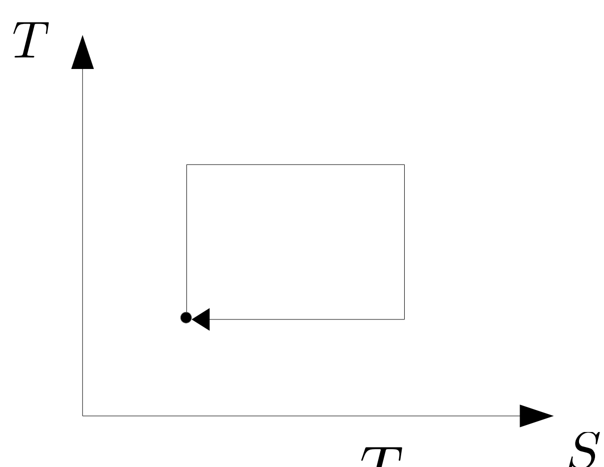


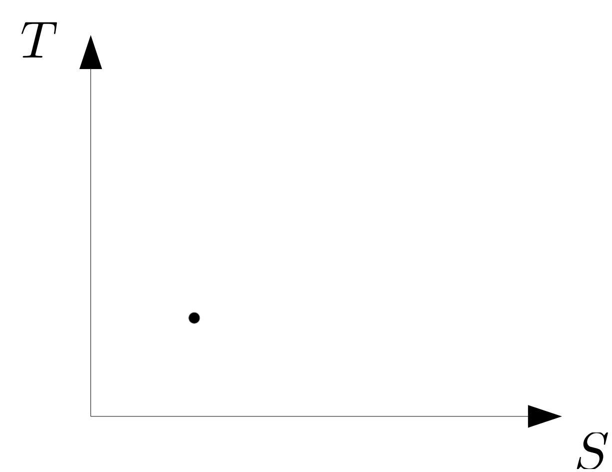


$$\eta = 1 - \frac{T_c}{T_h}$$

Limitations on the set of Hamiltonians. A toy model of the toy models.

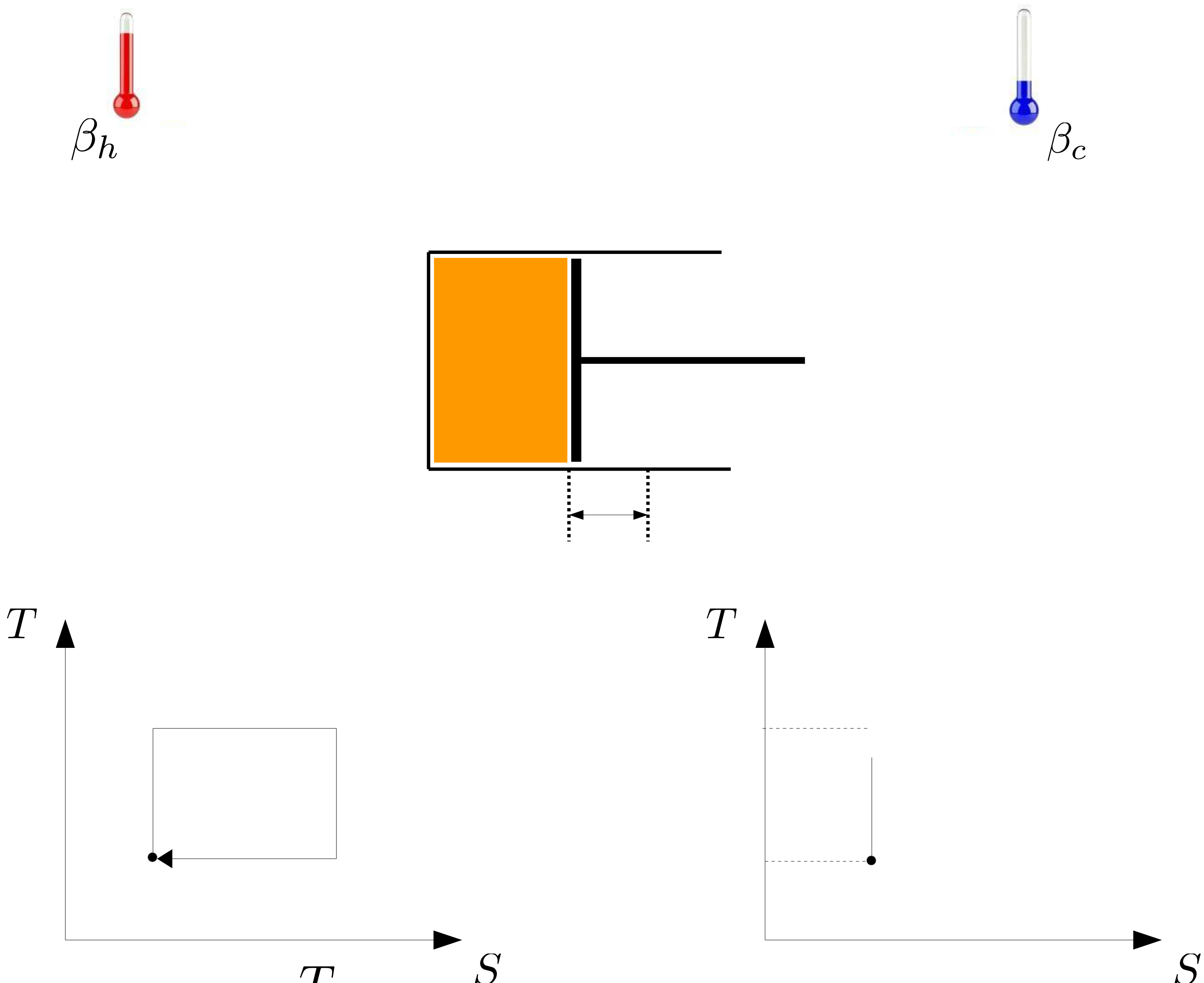






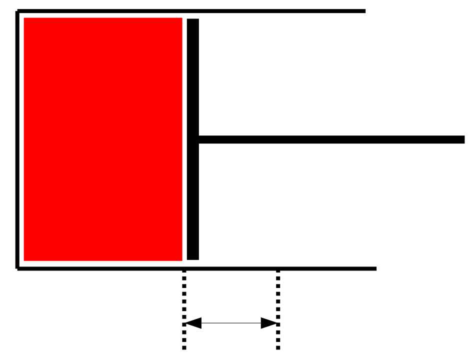
$$\eta = 1 - \frac{T_c}{T_h}$$

Limitations on the set of Hamiltonians. A toy model of the toy models.

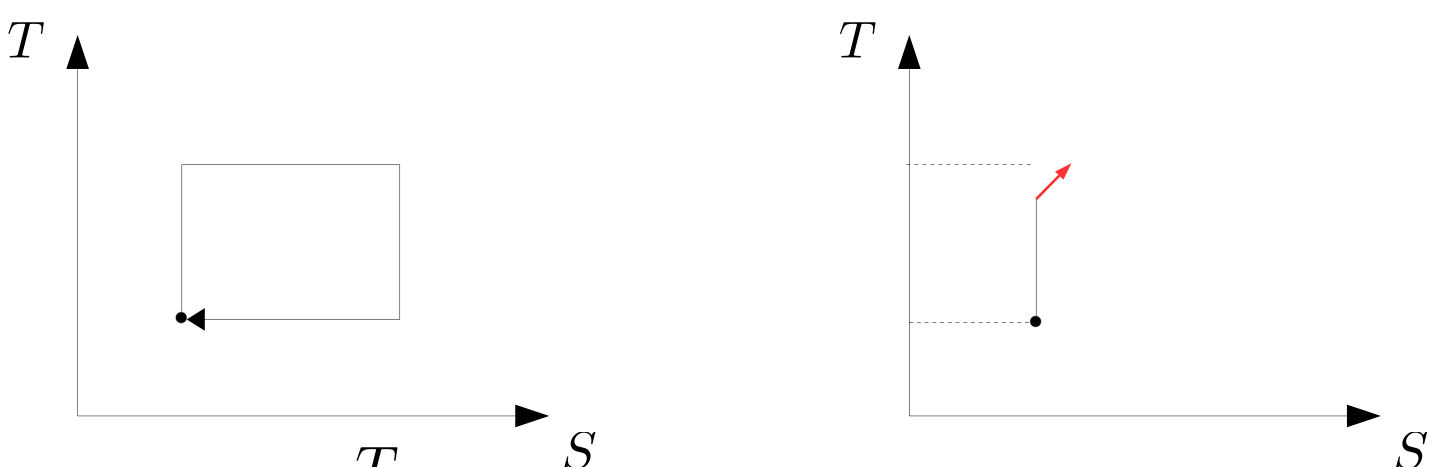


$$\eta = 1 - \frac{T_c}{T_h}$$

Limitations on the set of Hamiltonians. A toy model of the toy models.

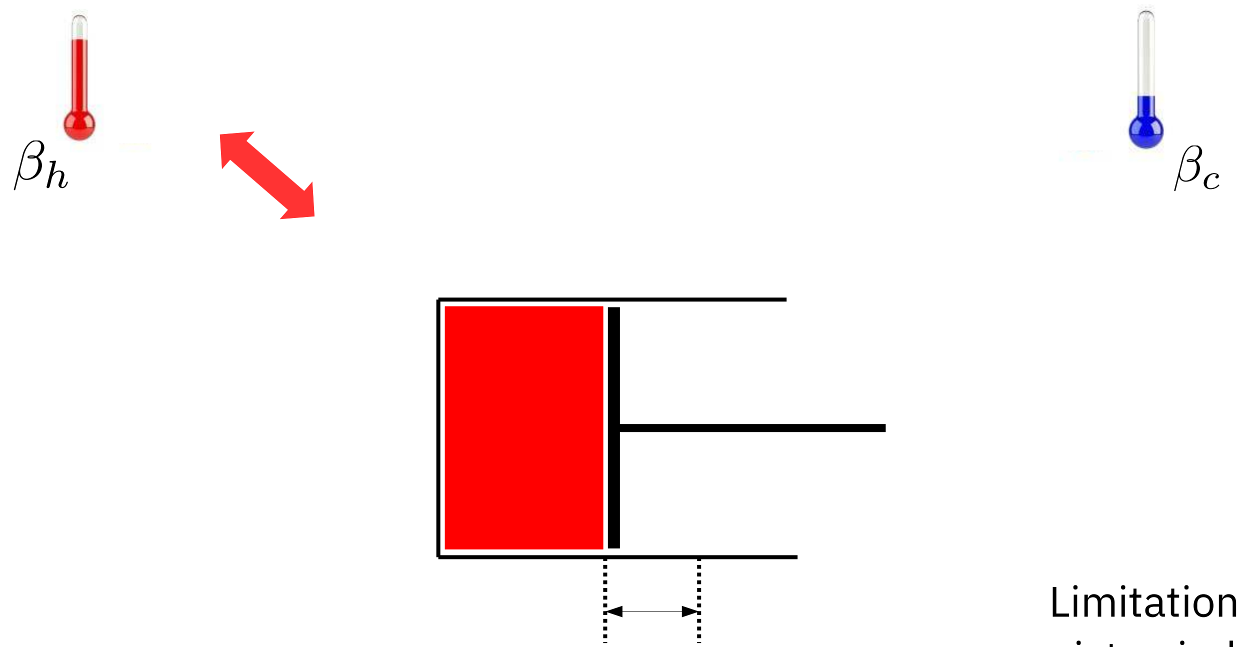


Limitations on the position of the piston induce irreversibility

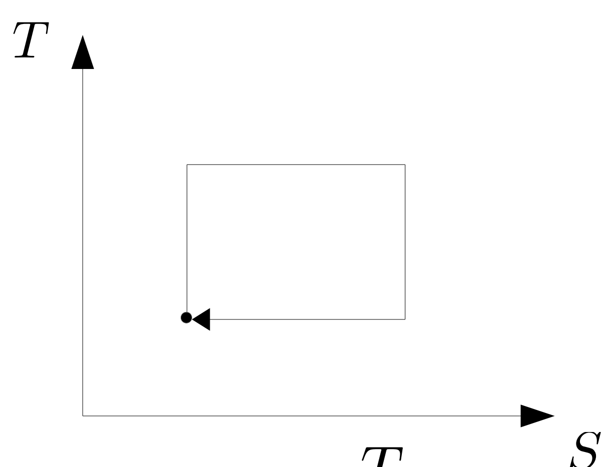


$$\eta = 1 - \frac{T_c}{T_h}$$

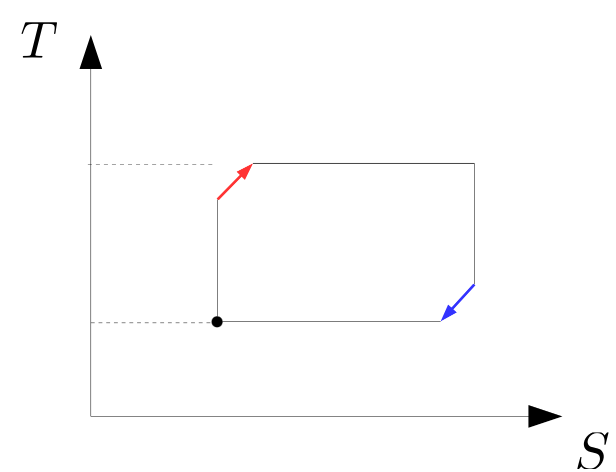
Limitations on the set of Hamiltonians. A toy model of the toy models.



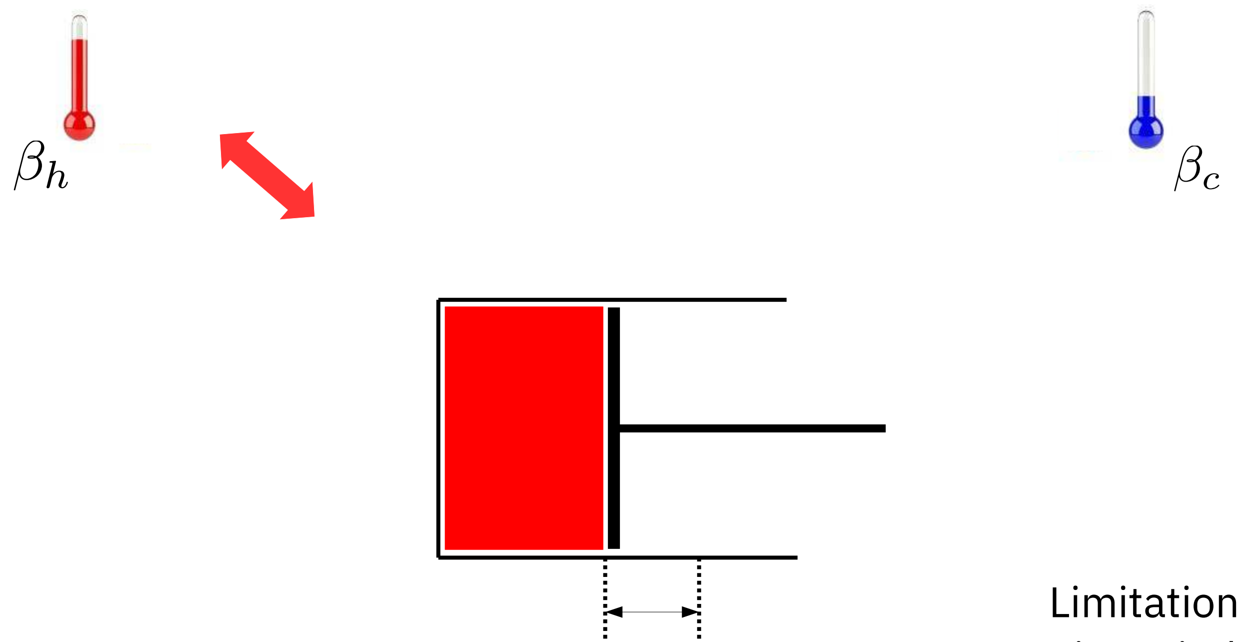
Limitations on the position of the piston induce irreversibility



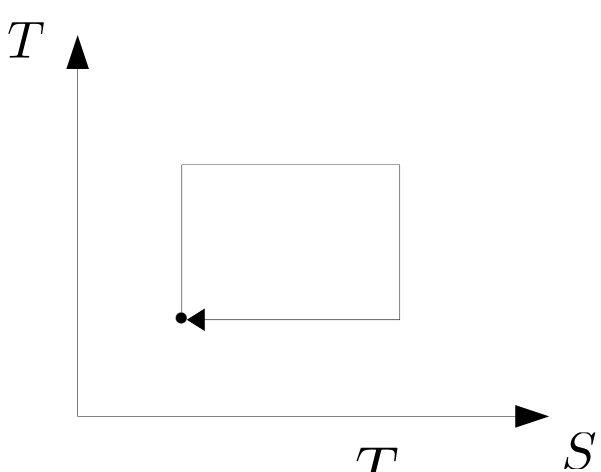
$$\eta = 1 - \frac{T_c}{T_h}$$



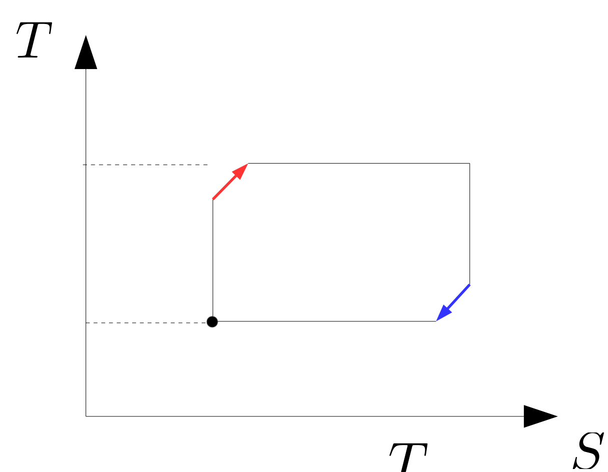
Limitations on the set of Hamiltonians. A toy model of the toy models.



Limitations on the position of the piston induce irreversibility



$$\eta = 1 - \frac{T_c}{T_h}$$



$$\eta < 1 - \frac{T_c}{T_h}$$

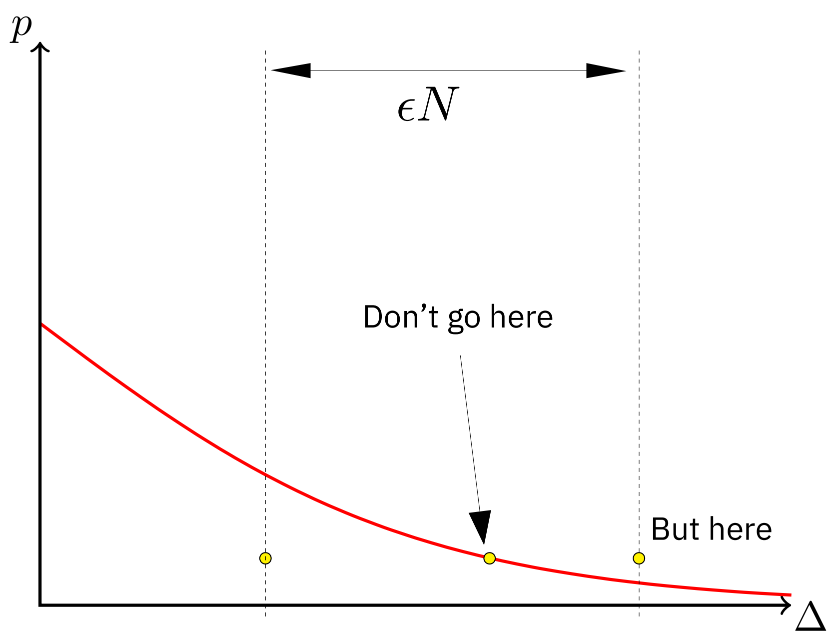
Ising model: Toy-model explanation of behaviour of ferromagnetic regime

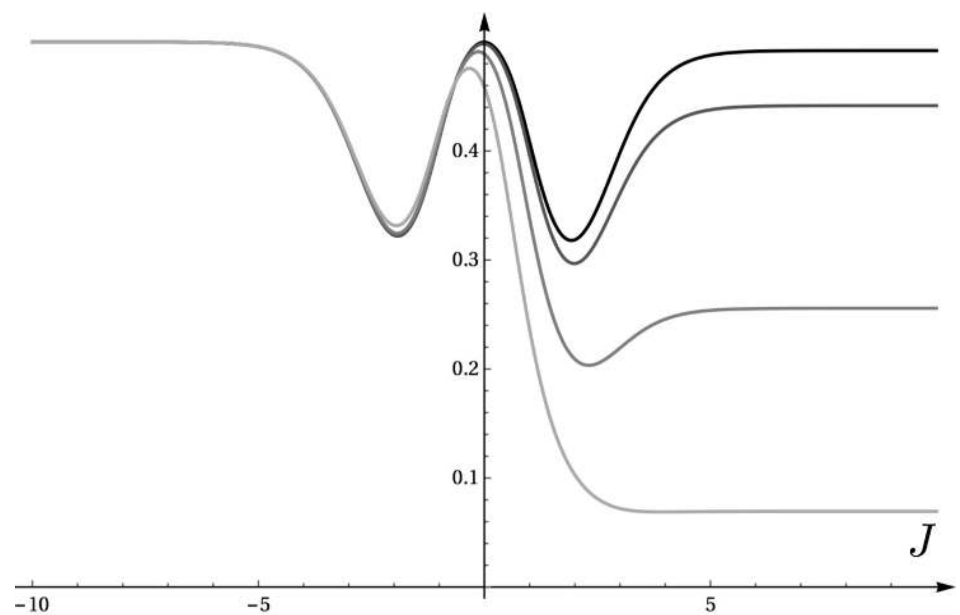
$$H = J\left[\frac{h(t)}{J}\sum_j \sigma_z^j - \sum_j \sigma_z^j\sigma_z^{j+1}\right] = JH_J$$

For finite N, but J extremely large, we have (for zero magnetic field)

$$\omega_\beta(H) = \omega_{J\beta}(H_J) \approx \frac{1}{2}\left[|0\ldots 0\rangle\langle 0\ldots 0| + |1\ldots 1\rangle\langle 1\ldots 1|\right]$$

For **small fields** but **very large J**, state is still essentially confined to |0...0> and |1....1>, but changing the field by a small amount ϵ changes the energy difference between the two states by a large amount (proportional to N). Thus, finite but small error on magnetic field yields huge errors in the conotrol of the state.





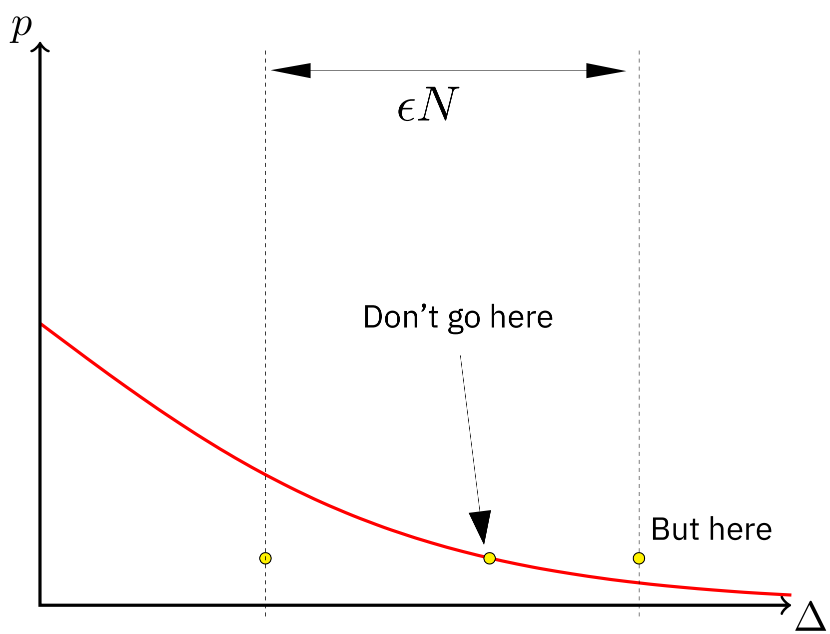
Efficiency at maximum work for 6 spins and different accuracies for magnetic field in cycle

$$H = J\left[\frac{h(t)}{J}\sum_j \sigma_z^j - \sum_j \sigma_z^j\sigma_z^{j+1}\right] = JH_J$$

For finite N, but J extremely large, we have (for zero magnetic field)

$$\omega_\beta(H) = \omega_{J\beta}(H_J) \approx \frac{1}{2}\left[|0\ldots0\rangle\langle0\ldots0| + |1\ldots1\rangle\langle1\ldots1|\right]$$

For **small fields** but **very large J**, state is still essentially confined to |0...0> and |1....1>, but changing the field by a small amount ϵ changes the energy difference between the two states by a large amount (proportional to N). Thus, finite but small error on magnetic field yields huge errors in the conotrol of the state.



$$\lim_{J\to\infty}\eta = \frac{T_h - T_c}{T_h}\mathcal{O}(\mathrm{e}^{-\epsilon N})$$

Work-density in cycle given by

$$\frac{W}{N} = (T_h - T_c)\frac{\Delta S}{N} - \text{corrections}$$

Required that we can choose magnetic fields in parts of thermodynamic cycle (as functions of J) so that we have finite **entropy density** for any value of J. For **large J**, this is **impossible** in the **ferromagnetic** regime, since groundstate degeneracy does not grow exponentially with N and the system is essentially stuck in the groundstate subspace of the interaction.

Ising model: Explanation for **anti-ferromagnetic** regime

$$H = |J| \left[-\frac{h}{|J|} \sum_j \sigma_z^j + \sum_j \sigma_z^j \sigma_z^{j+1} \right] = |J| H_{|J|}$$

For finite N, but |J| extremely large, we have

$$\omega_\beta(H) = \omega_{|J|\beta}(H_{|J|}) \approx \text{normalized ground-state projector of } H_{|J|}$$

For **h = 2|J|**, we have **exponential** ground-state degeneracy:

$$\text{Energy}(\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow) = [-2N + N] = -N$$

$$\text{Energy}(\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow) = [(-2N + 4) + (N - 4)] = -N$$

$$\text{Energy}(\downarrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow) = [(-2N + 8) + (N - 8)] = -N$$

Can choose at least for every second spin whether we want to flip or not:
Exponentially many choices

Finite entropy density in groundstate for h=2|J|.

Zero entropy density for h=0.

--> Finite work-density.