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Time–varying rational expectations models

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ABSTRACT

This paper develops a comprehensive theory for rational expectations models with time–varying (random) coefficients. Based on the Multiplicative Ergodic Theorem it develops a "linear algebra" in terms of Lyapunov exponents, defined as the asymptotic growth rates of trajectories. Together with their associated Lyapunov spaces they provide a perfect substitute for the eigenvalue/eigenspace analysis used in constant coefficient models. In particular, they allow the construction of explicit solution formulas similar to the standard case. These methods and their numerical implementation is illustrated using a canonical New Keynesian model with a time–varying policy rule and lagged endogenous variables.

1. Introduction

This paper develops a comprehensive theory for rational expectations models with time–varying (random) coefficients. Based on the Multiplicative Ergodic Theorem (MET) it develops a "linear algebra" in terms of Lyapunov exponents, defined as the asymptotic growth rates of trajectories. Together with their associated Lyapunov spaces they provide a perfect substitute for the eigenvalue/eigenspace analysis used in constant coefficient models. In particular, they allow the construction of explicit solution formulas similar to the case of constant coefficients.

The necessity for the introduction of Lyapunov exponents stems from the insight that the eigenvalues of the "time frozen" or "local" coefficient matrices provide in general no information about the stability of the underlying difference equation. Elaydi (2005, p. 191), Colonius and Kliemann (2014, pp. 109–110), Costa et al. (2005, section 3.3.2) and Neusser (2017, appendix A) provide examples where the coefficient matrix alternates between two alternatives such that the model becomes unstable although all eigenvalues in both alternatives are absolutely smaller than one.[2] Fortunately, the Lyapunov exponents are a perfect substitute. Taking a *random dynamical systems* perspective, Oseledets' celebrated Multiplicative Ergodic Theorem (MET) lifts the eigenvalue\eigenvector analysis used in the constant coefficient case to the case of stochastically varying coefficients using the Lyapunov exponents\spaces.[3] Thus, as shown in this paper, the MET paves the way for the derivation of explicit solution formulas for rational expectations model with stochastically varying coefficients. These solution formulas

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[2] Francq and Zakoïan (2001) provide further illuminating examples in the context of multivariate Markov-switching ARMA models.

[3] Colonius and Kliemann (2014) provides a clear and accessible presentation of the MET by relating it to the standard eigenvalue\eigenspace analysis. The monograph by Arnold (2003) and Viana (2014) also offer an elaborated and excellent expositions, but are mathematically more demanding.

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turn out to be in the spirit of Blanchard and Kahn (1980), Klein (2000), and Sims (2001) and are therefore directly interpretable in economic terms. They typically involve a forward looking component which can be interpreted as an expected present discounted value with stochastic factors.

The theory outlined in this paper is very versatile and readily implementable. It allows to analyze a wide class of rational expectations models which, so far, have been inaccessible or hardly accessible to economists. It encompasses, in particular, models with Markov–switching or autoregressively moving coefficients with and without lagged dependent variables. Thereby, the exogenous forcing variables follow general stochastic processes, including ARMA processes. There is, however, a price to pay for this generalization: the Lyapunov exponents and their associated Lyapunov spaces cannot, in general, be computed analytically, but are only accessible via numerical procedures. This alleged disadvantage is compensated by powerful numerical algorithms which do not only allow the computation of the Lyapunov exponents, but also their corresponding Lyapunov spaces (see Dieci and Elia, 2008 and Froyland et al., 2013 for details). In doing so, this is the first paper to make explicit use in economics of Lyapunov exponents/spaces along with the MET to analyze dynamic, respectively rational expectations models.

The stimulus to deal with time-varying coefficients in rational expectations models is driven from theoretical as well as empirical considerations. Indeed, there are several convincing reasons to believe in time-varying coefficients. First, time-varying coefficient models arise naturally from the linearization of nonlinear models along solution paths (Elaydi, 2005, p. 219–220). Second, the relationships describing the economy undergo structural changes giving rise to drifting coefficients as emphasized by Lucas' critique. Sargent (1999), for example, provides an interpretation in terms of self-confirming equilibria and learning. Third, policies and policy rules are subject to change. Cogley and Sargent (2005), Primiceri (2005), or Chen et al. (2015), among many others, provide empirical evidence with regard to U.S. monetary policy.[4] Moreover, Fernández-Villaverde and Rubio-Ramírez (2008) presented evidence and interpretations of drifting structural parameters in dynamic stochastic general equilibrium (DSGE) models.

Related Literature This paper shares the ambition expressed in the pioneering work by Farmer et al. (2009, 2011), and Foerster et al. (2016) to provide a solid and adequate methodology to analyze rational expectations models with time–varying coefficients. This venture triggered a number of papers which extended the original framework to incorporate notably lagged endogenous and predetermined variables (Barthélemy and Marx, 2017; Cho, 2016; Foerster, 2016). An important aspect of this literature is their concern for the derivation of criteria for model determinacy and stability. These criteria are based on a mean-square stability as originally proposed by Costa et al. (2005). As deserving and inspiring this research may be, the framework seems unnecessarily restrictive. First, the reliance on finite state Markov-switching mechanisms as a source of time-variation excludes many interesting alternatives, like slowly drifting coefficients, coefficients moving according to some autoregressive scheme, or even some nonlinear processes. As long as these processes are ergodic and possess a strictly stationary distribution, they would qualify in the framework presented here as a mechanism for time variation. Second, the focus on second moments may exclude some variables of interest to economists such as asset prices which may not have second moments. Finally, in contrast to the Lyapunov spectrum (the set of all Lyapunov exponents) the mean-square stability criterion and its adaptations do not completely capture the dynamic properties of the model. This becomes, however, necessary if one wants to derive and analyze explicit model solutions. This will require to divide the Lyapunov exponents and their associated Lyapunov spaces into those associated with stable, respectively explosive dynamics. The approach presented here will cope with these issues without substantial costs.

Davig and Leeper (2007) propose an alternative route for solving Markov-switching rational expectations models. Their method circumvents the nonlinearity induced by the presence of random coefficients by defining for each endogenous variable corresponding set of variables indexed by state. The advantage of inflating the number of variables is that the resulting state equation becomes linear with constant coefficients and can therefore be analyzed by standard methods. The disadvantage of this approach seems to be that it is restricted to relatively small models with a small number of states and that it is restricted to Markov-switching models. Moreover, as shown by Farmer et al. (2010) it does not capture the complete set of solutions.

There is a related, but apparently disconnected literature on autoregressive processes with coefficients given as strictly stationary ergodic processes. This literature establishes that a unique strictly stationary solution exists whenever the top–Lyapunov exponent (the largest Lyapunov exponent) is strictly negative (Bougerol and Picard, 1992; Brandt, 1986; Francq and Zakoïan, 2001). Similarly, in the context of Markov–switching multivariate ARMA models in the spirit of Hamilton (1989, 2016), Francq and Zakoïan (2001) derive a necessary and sufficient condition for the existence of a second–order stationary solution (i.e. for mean–square stability). Not surprisingly, their eigenvalue criterion is practically similar to those provided in Cho (2016) and Foerster (2016). More specifically, compare (Francq and Zakoïan, 2001, theorem 2) with Farmer et al. (2009, theorem 2). Although the top–Lyapunov exponent is an important characteristic, it does not in general capture all dynamic properties of the underlying model.

While it is true that the analysis of time-varying difference equations by Lyapunov exponents is not original, it is new to economists. The paper is, however, not just a simple transfer of technology. Applications in the natural sciences typically lack the forward looking aspect which is characteristic for rational expectations models. This specificity of economic models implies that it is not sufficient to examine just the top (largest) Lyapunov exponent as in the time series analysis papers

[4] For more empirical references see Davig and Leeper (2007, foonote 4).

above, but that it is necessary to consider the set of all Lyapunov exponents in relation to the number of predetermined variables. This is where Oseledets' theorem comes into play. It shows how to split the state space into subspaces of stable, respectively exploding dynamics. If the boundary conditions (initial conditions in the form of predetermined variables and boundedness) determine exactly one initial vector in the stable subspace, the model has a unique solution. The corresponding solution formula involves, in general, backward looking as well as forward looking components and is stated explicitly in the paper. If, on the contrary, the boundary conditions are not sufficient to pin down a unique initial vector in the stable subspace, but allow for several linearly independent initial vectors in the stable subspace, the model is indeterminate.

The approach via Lyapunov exponents/spaces is not in contradiction with those put forward by Famer et al. or Davig and Leeper (2007). In particular, Francq and Zakoïan (2001) have already established the correspondence between the condition that the top Lyapunov exponent (largest Lyapunov exponent) is strictly negative and the condition that all eigenvalue of a certain matrix are strictly smaller than one in absolute value.[5] This condition ensures the existence of a mean-square stable (second-order stationary). Thus, it is not surprising that the simulation results for the New Keynesian model with time-varying Taylor rule are in line with those reported in the literature.

The paper proceeds by first exposing the general setup in Section 2. It lays out the basic assumptions, presents and explains the Multiplicative Ergodic Theorem (MET). The main contribution of the paper is presented in Section 3 which derives explicit solution formulas together with some implications. Having presented the general theory, Section 4 illustrates the usefulness of the proposed methods by applying them to a simple New Keynesian model with a randomly switching Taylor rule. More specifically, I allow the policy to switch between two states which would lead to determinate, respectively indeterminate models in the deterministic case. The economic relevance and the consequences of these two alternative rules have been analyzed by Galí (2011). This model also serves as a prime example in the contributions cited above. Finally, I draw some conclusions for further applications and research. Additional details and explanations are provided in several appendices.

2. Random coefficients rational expectations models

2.1. Model setup

The class of rational expectations models with time-varying (random) coefficients analyzed in this paper consists of an affine state equation which describes the evolution of the state $x_t \in \mathbb{R}^d$ over time subject to boundary conditions. More specifically, the state equation takes the following form:

$$\mathbb{E}_t x_{t+1} = \psi_t(x_t) = A_t x_t + b_t, \qquad t \in \mathbb{Z}, \tag{2.1}$$

where ψ_t is a randomly chosen affine map. In contrast to conventional rational expectations models, the coefficient matrix A_t is not constant, but is allowed to vary randomly over time. For convenience and to avoid unnecessary technical intricacy, $A_t \in \mathbb{GL}(d)$, the set of nonsingular $(d \times d)$ matrices, for all t. The term $b_t \in \mathbb{R}^d$ captures all exogenous forces or shocks which impinge on the economy. If the coefficient matrix A_t is constant, the expectational difference Eq. (2.1) encompasses most of the rational expectations models encountered in the literature. Their properties have been exhaustively analyzed in Blanchard and Kahn (1980), Klein (2000) and Sims (2001) to name just the most relevant contributions.

The above setup is quite general and versatile. As long as we stick to invertible matrices, it also encompasses state equation of the type

$$B_{1,t} y_t = \mathbb{E}_t (B_{2,t+1} y_{t+1}) + z_t$$

by making the change of variable $x_t = B_{2,t} y_t$. In this case $A_t = B_{1,t} B_{2,t}^{-1}$ and $b_t = -z_t$. Similarly, for equations of the type

$$B_{1,t} x_t = B_{2,t} \mathbb{E}_t x_{t+1} + z_t$$

if we define $A_t = B_{2,t}^{-1} B_{1,t}$ and $b_t = -B_{2,t}^{-1} z_t$. We can also treat models with lagged endogenous variables which have proven technically difficult to analyze so far (Cho, 2016). Let, for example,

$$\mathbb{E}_t y_{t+1} = B_{1,t} y_t + B_{2,t} y_{t-1} + b_t.$$

By enlarging the state space and using the companion form the above equation can be written in the format of Eq. (2.1):

$$\mathbb{E}_t x_{t+1} = \mathbb{E}_t \begin{pmatrix} y_{t+1} \\ y_t \end{pmatrix} = \underbrace{\begin{pmatrix} B_{1,t} & B_{2,t} \\ I_d & 0 \end{pmatrix}}_{=A_t} \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} b_t \\ 0 \end{pmatrix}.$$

Also the process governing the evolution of A_t can be quite involved. For example, regime-switching and autoregressive schemes are easily implemented. The only practically important feature is that the evolution of A_t is exogenous to the model, i.e. determined outside the model, and that it is taken into account in forming conditional expectations. In the

[5] This matrix is quoted as equation (18) in Farmer et al. (2009) and as matrix $\widetilde{P}$ on page 345 in Francq and Zakoïan (2009).

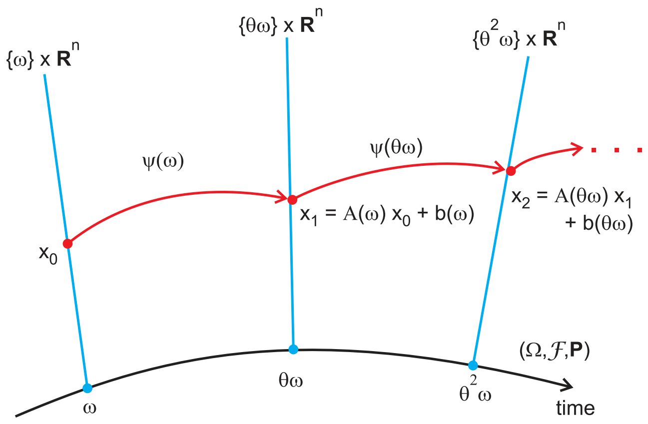

Fig. 1. The Evolution of an Affine Random Dynamic System.

example treated in Section 4 the evolution of A_t is governed by a hidden Markov chain which makes the model a Markov-switching rational expectations model. A similar remark can be made with respect to b_t. In particular, b_t can follow some autoregressive process.

From a conceptual point of view it is important to have a clear understanding on the randomness driving the system. All random variables are defined with respect to a *common* probability space $(\Omega, \mathfrak{F}, \mathbf{P})$, i.e. a sample space (set of all possible outcomes) Ω endowed with a σ–algebra $\mathfrak{F}$ and a probability measure $\mathbf{P}$. To highlight the dependence on a particular outcome $\omega \in \Omega$, we sometimes write $A_t(\omega)$, respectively $b_t(\omega)$.

Define $\mathfrak{F}_t$ as $\mathfrak{F}_t = \sigma\{(x_s, A_s, b_s) : s \leq t\}$, i.e. $\mathfrak{F}_t$ is the smallest σ–algebra such that (x_s, A_s, b_s) is measurable for all $s \leq t$. The sequence of σ–algebras $\mathfrak{F}_t$ then becomes a filtration adapted to $\{x_t\}$ and $\{(A_t, b_t)\}$ with $\mathfrak{F}_t \subseteq \mathfrak{F}$. $\mathbb{E}_t x_{t+1}$ denotes the conditional expectation with respect to $\mathfrak{F}_t$, i.e. $\mathbb{E}_t x_{t+1} = \mathbb{E}[x_{t+1} \mid \mathfrak{F}_t]$. This implies that in forming their expectations agents take not only the current and past evolution of x_{t-s} and b_{t-s}, but also of A_{t-s} into account, $s \leq t$. This despite the fact the A_t is determined outside the model. In this sense, depending on the concrete economic issue, some models may be regarded as being incomplete because the randomness driving A_t should have already been accounted for in the underlying optimization problem of agents. This may, however, be too demanding so that models with random coefficients are still a very beneficial modeling device even in such settings.

At this point it is useful to introduce the notion of the *time shift operator* θ: $\Omega \to \Omega$. If $A_t(\omega)$ is a particular sample path, the time shifted realization $A_t(\theta\omega)$ is defined to be $A_{t+1}(\omega)$. The s-fold application of θ, denoted by θ^s, then is $A_t(\theta^s\omega) = A_{t+s}(\omega)$. In accordance with this notation $\theta^0 = \mathrm{Id}$ and, because the time index t runs over the set of integers $\mathbb{Z}$, we also have $\theta^r\theta^s = \theta^{r+s}$ for all $r, s \in \mathbb{Z}$. This implies that θ is invertible. In addition, it can be shown that θ is measure preserving, i.e. $\theta\mathbf{P} = \mathbf{P}$, and that θ is ergodic (see f.e. Coudène, 2016, chapter 2).[6] In this vein, we denote $A_t(\omega)$ by $A(\theta^t\omega)$. The same notation is also applied to b_t and ψ_t. In Section 4 where we will analyze a Markov-switching model, we will be more specific.

Following Arnold (2003, chapter 1), the evolution of the system on the bundle $\Omega \times \mathbb{R}^n$ can be envisioned as in Fig. 1. While ω is shifted by θ to $\theta\omega$, the point x_0 in the fiber $\omega \times \mathbb{R}^d$ is shifted to $x_1 = \psi(\omega)x_0 = A(\omega)x_0 + b(\omega)$ in the fiber $\theta\omega \times \mathbb{R}^d$. In the next period $\theta\omega$ is shifted to $\theta^2\omega$ whereas x_1 is shifted to $x_2 = \psi(\theta\omega) = A(\theta\omega) + b(\theta\omega)$ and so on. Thus, on each fiber the system is affine in the usual sense.

For technical reasons, the paper assumes the following integrability condition.

Assumption 1 (Integrability)**.**

$$\log^+ \|A\|, \quad \log^+ \|A^{-1}\| \quad \text{and} \quad \log^+ \|b\| \in \mathrm{L}^1(\Omega, \mathfrak{F}, \mathbf{P}).$$

Thereby $\log^+ x$ stands for $\max\{\log x, 0\}$. This is, from a practical point of view, a very weak condition which allow $\{A_t\}$ and $\{b_t\}$ to follow a wide variety of stochastic processes. The condition is, for example, satisfied for first order stationary processes. Indeed if $\{b_t\}$ is first order stationary, meaning that $\mathbb{E}b_t$ exists and is constant, Jensen's inequality implies that $\mathbb{E}\log\|b_t\| \leq \log\mathbb{E}\|b_t\| < \infty$. Hence the integrability condition is satisfied. The argument would hold even more so if $\{b_t\}$ is a covariance stationary process (the typical assumption made in economics with the respect to shock processes). While the integrability condition for A, respectively A^{-1}, is needed for the MET to apply, the assumption with regard to b becomes only relevant when computing the forward component of the particular solution in Theorem 2. The reason for this is that this forward component involves the expectation operator.

Here and in the following $\|.\|$ denotes the operator norm induced by the Euclidean metric, i.e. $\|A\| = \max_{\|x\|=1} \|Ax\| = \delta_1$ where δ_1 is the largest singular value of A, i.e. δ_1 is the positive square root of the largest eigenvalue of $A'A$. Because all

[6] Maps with these properties are called *ergodic metric dynamical systems* in the corresponding literature (Arnold, 2003, Appendix A).

norms are equivalent in $\mathbb{R}^d$, the integrability assumption and the Multiplicative Ergodic Theorem (MET) presented below are independent from any specific submultiplicative norm.

Boundary conditions The model setup is completed by assuming some boundary conditions. They usually come in two forms: initial value and boundedness conditions. The former can be written as

$$c = Rx_0, \qquad c \in \mathbb{R}^r, \tag{2.2}$$

where R is a given $(r \times d)$ matrix of rank r, $0 \leq r \leq d$. In its simplest and most widely used form $r \leq d$ variables, say the first r variables, are predetermined variables. In this case $R = (I_r, 0)$ which fixes the first r elements of x_0 to be equal to c. When $r < d$, the initial value conditions are not sufficient to pin down x_0 uniquely. Hence, they are complemented by boundedness conditions:

$$\text{there exists } M \in \mathbb{R} \text{ such that } \|x_t\| < M \text{ for all } t \in \mathbb{Z}. \tag{2.3}$$

The latter conditions are usually rationalized by assuming b_t to be bounded which, depending on the particular model in mind, implies that unbounded solutions $\{x_t\}$ are economically not feasible or sensible.

2.2. Preliminary considerations

The expectational difference Eq. (2.1) satisfies the *superposition principle*: given two solutions $\{x_t^{(1)}\}$ and $\{x_t^{(2)}\}$, then $\{x_t^{(1)} - x_t^{(2)}\}$ satisfies the linear expectational difference

$$\mathbb{E}_t x_{t+1} = A_t x_t. \tag{2.4}$$

Thus, every solution is of the form:

$$x_t = x_t^{(g)} + x_t^{(p)}$$

where $\{x_t^{(g)}\}$ denotes the general solution of the linear Eq. (2.4) and $\{x_t^{(p)}\}$ a particular solution of Eq. (2.1). Hence, the solution can be found in two steps. First, find the general solution of the linear Eq. (2.4) and then look for a particular solution of Eq. (2.1).[7] We first turn our attention to the general solution of the linear Eq. (2.4) and postpone the construction of a particular solution to Section 3.

In order to find the general solution to the linear expectational difference Eq. (2.4) define $\Phi(t) = \Phi(t, \omega)$ as the random matrix product:

$$\Phi(t) = \Phi(t,\omega) = \begin{cases} A_{t-1}(\omega)\dots A_1(\omega)A_0(\omega), & t = 1, 2, \dots; \\ I_d, & t = 0; \\ A_t(\omega)^{-1}\dots A_{-1}(\omega)^{-1}, & t = -1, -2, \dots \end{cases}$$

This defines a *linear cocycle* over θ, i.e. $A : \Omega \to \mathbb{GL}(n)$ is measurable, $\Phi(0, \omega) = I_d$, and $\Phi(t + s, \omega) = \Phi(t, \theta^s\omega)\Phi(s, \omega)$ for all $t, s \in \mathbb{Z}$ and all $\omega \in \Omega$. Hence, $I_d = \Phi(0, \omega) = \Phi(t, \theta^{-t}\omega)\Phi(-t, \omega)$ which implies the following Lemma by direct calculation.

Lemma 1. *The cocycle properties imply*

$$\Phi(-t,\omega) = \Phi(t, \theta^{-t}\omega)^{-1}$$
$$\Phi(t,\omega)^{-1} = \Phi(-t, \theta^{t}\omega).$$

Next define a new variable y_t as $y_t = \Phi(t)^{-1}x_t$. It is easy to see that $\{y_t\}$ is a martingale:

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t\left(\Phi(t+1)^{-1}x_{t+1}\right) = \Phi(t+1)^{-1}\mathbb{E}_t x_{t+1} = \Phi(t+1)^{-1}A_t x_t = y_t.$$

Similarly, the time reversed process $\tilde{y}_t = y_{-t}$, $t \in \mathbb{Z}$, is also a martingale. This implies without any additional assumptions that there exists a random variable y satisfying $\lim_{t\to\infty} y_t = y$ a.s. and in mean (see Grimmett and Stirzaker, 2001, section 12.7). Moreover, the original martingale can be reconstructed from y by setting $y_t = \mathbb{E}(y \mid \mathfrak{F}_t)$. Thus, the space of martingales can be continuously parameterized by the space of random variables which are measurable with respect to $\mathfrak{F}$ where $\mathfrak{F} = \sigma\left(\bigcup_{t\in\mathbb{Z}}\mathfrak{F}_t\right)$.[8] The general solution of the linear Eq. (2.4) can therefore be represented as

$$x_t = (A_{t-1}\dots A_1 A_0)x = \Phi(t)x, \tag{2.5}$$

where x is some random variable measurable with respect $\mathfrak{F}$.

Given some starting value x and a realisation ω, the Eq. (2.5) determines one particular trajectory or solution. Hence, trajectories are parameterized by x and ω and are denote by $x_t = \varphi(t, \omega, x) = \Phi(t, \omega)x$. The existence and the stability properties of the solutions (2.5) depend crucially on the convergence of the matrix products $\Phi(t)$.[9]

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[7] Cho (2016) follows a similar, but not exactly equal, two step procedure.

[8] Compare this to Klein (2000, Definition 4.3 and Assumption 4.2)

[9] The study of random matrix products has a long history going back to Bellman (1954) and culminated in the acclaimed theorems by Furstenberg and Kesten (1960) and, more relevant for this paper, Oseledets' Multiplicative Ergodic Theorem (MET) (Oseledets, 1968).

2.3. Definition and interpretation of Lyapunov exponents

The convergence behavior of $\Phi(t)$ is best understood via the notion of *Lyapunov exponents*. As a motivation consider first the univariate case $d=1$. In this case $\Phi(t)$ is just a product of randomly drawn nonzero scalars a_t. Thus, $\Phi(t)=a_{t-1}\ldots a_1 a_0$. Given some trajectory $x_t=\varphi(t,\omega,x)=\Phi(t)x$, consider a new trajectory obtained by a small perturbation Δx of the starting value x. The resulting change Δx_t at time t in the trajectory is then $\Delta x_t=\varphi(t,\omega,\Delta x)=\Phi(t,\omega)\Delta x$. This motivates to define the *Lyapunov exponent* λ as the mean exponential rate of divergence or convergence of the two trajectories as $t\to\infty$ and $|\Delta x|\to 0$. Hence λ approximately satisfies $|\Delta x_t|\approx e^{t\lambda}|\Delta x|$ and the Lyapunov exponent can be expressed as

$$\lambda(\omega,x)=\lim_{t\to\infty}\lim_{|\Delta x|\to 0}\frac{1}{t}\log\frac{|\Delta x_t|}{|\Delta x|}=\lim_{t\to\infty}\frac{1}{t}\sum_{j=0}^{t-1}\log|a_j|.$$

In order to guarantee that the limits make sense, the limes superior is usually used instead. The definition above implies that convergence is achieved when the Lyapunov exponent is negative and divergence when it is positive.[10] Hence, the Lyapunov exponent characterizes the stability of the zero solution $x_t=\varphi(t,\omega,0)=0$, for all t, which represents the deterministic steady state. Moreover, if the sequence $a_t=a(\theta^t\omega)$ is ergodic and $\mathbb{E}\log|a|$ exists, Birkhoff's ergodic theorem implies $\lim_{t\to\infty}\frac{1}{t}\sum_{j=0}^{t-1}\log|a_j|=\mathbb{E}\log|a|$.[11]

In the multivariate case we have to make two modifications to the above definition. First, the absolute value is replaced by the norm. Second, the logarithm of a matrix product is not the sum of the logarithms of its factors. These considerations lead to define the Lyapunov exponents as follows:

$$\lambda(\omega,x)=\limsup_{t\to\infty}\frac{1}{t}\log\|\varphi(t,\omega,x)\|=\limsup_{t\to\infty}\frac{1}{t}\log\|\Phi(t,\omega)x\|,\qquad x\neq 0. \tag{2.6}$$

The Lyapunov exponent therefore describes the asymptotic exponential growth rate of the linear random dynamical system $x_{t+1}=A_t x_t$ with initial value $x\neq 0$. Readers not familiar with Lyapunov exponents should consult Appendix A which shows the relation to the eigenvalues in the case of constant coefficient.

The Lyapunov exponents can also be deduced from the eigenvalues of $\Phi(t)'\Phi(t)$. Because this matrix is positive definite, its eigenvalues $\mu_i(t)$, $i=1,\ldots,d$, are strictly positive and it can be diagonalized by an orthogonal matrix $Q(t)=(q_1(t),\ldots,q_d(t))$. Taking $x=q_i(t)$,

$$\frac{1}{t}\log\|\varphi(t,\omega,x)\|=\frac{1}{t}\log(q_i(t)'\Phi(t)'\Phi(t)q_i(t))^{1/2}=\frac{1}{2t}\log(q_i(t)'q_i(t)\mu_i(t))=\frac{1}{2t}\log(\mu_i(t)).$$

Thus, for large t, the Lyapunov exponent $\lambda(\omega,q_i)\approx\frac{1}{2t}\log(\mu_i(t))$. The derivation shows that there are at most d different Lyapunov exponents. Moreover, if x is arbitrary, the behavior is dominated by the direction corresponding to the largest eigenvalue.

It will be the subject of Oseledets' theorem to show that there exist only a finite number $\ell\leq d$ of different Lyapunov exponents. The set of all Lyapunov exponents is called the *Lyapunov spectrum*. These exponents are actually obtained as (double-sided) limits and are independent of x and ω. However, which one of the ℓ exponents is actually achieved in the limit depends on the initial direction x. The importance of the Lyapunov exponents with their corresponding Lyapunov spaces (introduced in the next section) derives from their ability to perfectly substitute the eigenvalue/eigenspace analysis which provides no information in the case of time-varying coefficients. Indeed, as noted in the Introduction, the eigenvalues of the "time frozen" or "local" matrices A_t are uninformative about the stability of the underlying system of difference equations.[12]

2.4. Oseledets' Multiplicative Ergodic Theorem

Indeed Oseledets' Multiplicative Ergodic Theorem (MET) (Oseledets, 1968) lifts the results from standard linear algebra which underlies the constant coefficient case to dynamical systems with random coefficients. An extensive exposition of this theorem with proofs and technical details can be found in Arnold (2003) and Viana (2014).[13] Here I follow Colonius and Kliemann (2014) and present a more accessible version.[14]

Theorem 1 (Multiplicative Ergodic Theorem (MET)). *Assuming the integrability condition 1 to hold, the linear random coefficient dynamical system $x_{t+1}=A_t x_t$ induces a splitting of $\mathbb{R}^d$ into $\ell\leq d$ linear subspaces $L_j(\omega)$, $j=1,\ldots,\ell$. These subspaces have the following properties:*

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[10] The case of a zero Lyapunov exponent is more intricate and will be excluded later on (see Assumption 2).

[11] See also Appendix B.

[12] See also footnote 1 for references.

[13] Argyris et al. (2015, Section 5.4) provides an intuitive introduction to the subject.

[14] It is not the most general formulation as there are versions of this theorem with one-sided time (i.e. $\mathbb{N}$ instead of $\mathbb{Z}$), continuous time, possibly non-invertible matrices A_t, and more general notions of time shift θ.

(i) *There is a decomposition (splitting)*

$$\mathbb{R}^d = L_1(\omega) \oplus \cdots \oplus L_\ell(\omega)$$

of $\mathbb{R}^d$ into the direct sum of ℓ random subspaces $L_j(\omega)$. These subspaces are not constant, but depend measurably on ω. However, their dimensions remain constant and equal to d_j. The spaces $L_j(\omega)$ are called Lyapunov spaces.

(ii) *The Lyapunov spaces are equivariant, i.e. $A(\omega)L_j(\omega) = L_j(\theta\omega)$.*

(iii) *There are real numbers $\infty > \lambda_1 > \ldots > \lambda_\ell > -\infty$ such that for each $x \in \mathbb{R}^d \setminus \{0\}$ the Lyapunov exponent $\lambda(\omega, x) \in \{\lambda_1, \ldots, \lambda_\ell\}$ exists as a limit and*

$$\lambda(\omega, x) = \lim_{t\to\pm\infty} \frac{1}{t} \log \|\varphi(t, \omega, x)\| = \lambda_j \text{ if and only if } x \in L_j(\omega) \setminus \{0\}.$$

(iv) *The limit*

$$\Upsilon(\omega) = \lim_{t\to\infty} \left(\Phi(t, \omega)'\Phi(t, \omega)\right)^{1/2t} \tag{2.7}$$

exists as a positive definite matrix. The different eigenvalues of $\Upsilon(\omega)$ are constants and can be written as $\exp(\lambda_1) > \ldots > \exp(\lambda_\ell)$; the corresponding random eigenspaces are $L_1(\omega), \ldots, L_\ell(\omega)$.

(v) *The Lyapunov exponents are obtained as limits from the singular values σ_k of $\Phi(t, \omega)$ as follows. The set of indices $\{1, 2, \ldots, d\}$ can be decomposed into subsets S_j, $j = 1, \ldots, \ell$, such that for all $k \in S_j$,*

$$\lambda_j = \lim_{t\to\infty} \frac{1}{t} \log \delta_k(\Phi(t, \omega)).$$

It is worth emphasizing that, although the Lyapunov subspaces $L_j(\omega)$ are random as they depend on ω, their dimension remains constant and equal to d_j. Moreover, they are, in general, not orthogonal to each other. Note also the equivariance (invariance or covariance) of these subspaces, i.e. $A(\omega)L_j(\omega) = L_j(\theta\omega)$. This property becomes relevant in the numerical implementation. There exists an alternative decomposition of $\mathbb{R}^d$ into orthogonal subspaces. These subspaces are, however, no longer equivariant (see Froyland et al., 2013, for details). Appendix B provides further explanations of the MET.

3. Construction of solutions

Given these preliminaries, it is now possible to construct explicit solution formulas. Divide, for this purpose, the state space into three subspaces $L^s(\omega)$, $L^c(\omega)$, and $L^u(\omega)$ corresponding to the Lyapunov spaces with negative, zero, and positive Lyapunov exponents:

$$L^s(\omega) = \bigoplus_{\lambda_j<0} L(\lambda_j, \omega), \quad L^c(\omega) = L(0, \omega), \text{ and } L^u(\omega) = \bigoplus_{\lambda_j>0} L(\lambda_j, \omega).$$

These subspaces are called the stable subspace, the center, and the unstable subspace, respectively. Thus, the zero solution of $x_{t+1} = A_t x_t$ is *asymptotically stable* if and only if all Lyapunov exponents are negative, or equivalently if the top Lyapunov exponent (the largest Lyapunov exponent) is negative (Bougerol and Picard, 1992; Brandt, 1986; Francq and Zakoïan, 2001). This is equivalent to $L^s(\omega) = \mathbb{R}^d$ for some (hence for all) ω. In the context of rational expectations models, however, the Lyapunov spectrum (the set of all Lyapunov exponents) is needed to derive an explicit solution formula.

The difference Eq. (2.1) is called *hyperbolic* if $L^c(\omega) = \emptyset$ or, equivalently, if all Lyapunov exponents are different from zero. For a hyperbolic difference equation the zero solution is called a *saddle point* if both $L^s(\omega)$ and $L^u(\omega)$ have dimensions $d^s = \dim L^s(\omega)$, respectively $d^u = \dim L^u(\omega)$, strictly greater than zero. In the following, we assume the difference equation to be hyperbolic.

Assumption 2 (Hyperbolicity)**.** The linear state Eq. (2.4) is hyperbolic, i.e. $L^c = \emptyset$.

The hyperbolicity assumption becomes particularly relevant when the difference equation is obtained from the linearization of a nonlinear system. Then the Hartman–Grobman theorem tells us that, under the assumption of hyperbolicity, the stability properties of the nonlinear system can be inferred from those of the linearized one (see f.e Robinson (1999, chapter 5) or Coudène (2016, chapter 8) and Arnold (2003, section 4.2.1 and chapter 7) for the random coefficient case). The hyperbolicity assumption also becomes relevant when constructing a particular solution.

Next define $\pi^s(\omega) : \mathbb{R}^d \to L^s(\omega)$ as the projection onto $L^s(\omega)$ along $L^u(\omega)$ and $\pi^u(\omega) : \mathbb{R}^d \to L^u(\omega)$ as the projection onto $L^u(\omega)$ along $L^s(\omega)$. These projections depend on ω because the Lyapunov spaces are random. They can be written in terms of matrices (Meyer, 2000, chapter 2.9):

$$\pi^s(\omega) = B(\omega)\begin{pmatrix} I_{d^s} & 0 \\ 0 & 0 \end{pmatrix} B^{-1}(\omega) \quad \text{and} \quad \pi^u(\omega) = B(\omega)\begin{pmatrix} 0 & 0 \\ 0 & I_{d^u} \end{pmatrix} B^{-1}(\omega)$$

where $B(\omega)$ is a basis of $\mathbb{R}^d$ obtained from the union of the basis of $L^s(\omega)$ and $L^u(\omega)$. Although the Lyapunov spaces vary implying varying basis $B(\omega)$, the dimensions d^s and d^u are fixed as stated by the MET.[15] Moreover, the equivariance of the Lyapunov spaces (see (ii) in Theorem 1) implies that $\pi^s(\theta^t\omega)\Phi(t, \omega) = \Phi(t, \omega)\pi^s(\omega)$ and similarly for $\pi^u(\omega)$.

[15] Compare this to the case of triangular matrices discussed in Appendix B.

Given these preliminaries, it is possible to construct a particular solution $x_t^{(p)}$ using the towering property of conditional expectations. This leads to the following theorem which is similar in spirit to the analysis of Blanchard and Kahn (1980), Klein (2000) or Sims (2001).

Theorem 2 (Solution Formula)**.** *The rational expectations model consisting of the difference Eq. (2.1) and the boundary conditions (2.2) and (2.3) admits a unique solution of the form*

$$x_t(\omega) = \Phi(t,\omega)x(\omega) + \underbrace{x_t^{(b)}(\omega) + x_t^{(f)}(\omega)}_{=x_t^{(p)}} \tag{3.1}$$

where

$$\begin{aligned} x_t^{(b)}(\omega) &= \quad \Phi(t,\omega)\sum_{j=0}^{\infty}\Phi(t-j,\omega)^{-1}\pi^s(\theta^{t-j}\omega)b_{t-j-1} \\ x_t^{(f)}(\omega) &= -\Phi(t,\omega)\,\mathbb{E}_t\left[\sum_{j=0}^{\infty}\Phi(t+j+1,\omega)^{-1}\,\pi^u(\theta^{t+j+1}\omega)b_{t+j}\right] \end{aligned}$$

if Assumptions 1 and 2 hold and if the rank condition

$$\operatorname{rank}\begin{pmatrix} R \\ (0 \quad I_{d^u})B(\omega)^{-1}\end{pmatrix} = d \tag{3.2}$$

is satisfied. In this case, $x(\omega)$ is uniquely determined from the equation system $c - Rx_0^{(p)} = Rx(\omega)$ and $\pi^u(\omega)x(\omega) = 0$.

Proof. The proof is relegated to Appendix C. □

As in the constant coefficient case, the particular solution $x_t^{(p)}$ is the sum of a backward solution $x_t^{(b)}$ and a forward solution $x_t^{(f)}$. Whereas $x_t^{(b)}$ corresponds to the negative Lyapunov exponents, $x_t^{(f)}$ is associated with the positive Lyapunov exponents. The two parts, therefore, have the interpretation of present values with time-varying "discount factors". The projections applied to b_{t-j-1} and b_{t+j} guarantee that discounting is applied only in the directions where the infinite sum converges.

The condition $\pi^u(\omega)x(\omega) = 0$ implies that the initial value $x(\omega)$ points into the direction of the stable subspace. Hence, the term $\Phi(t, \omega)x(\omega)$ vanishes in the long run and the system approaches the "moving steady state" $x_t^{(p)} = x_t^{(b)} + x_t^{(f)}$. This steady state is composed in general of backward looking and a forward looking component and induces an invariant distribution.

Corollary 1. *$r = d^s$ is a necessary condition for the existence of a unique nonexplosive solution.*

If $r < d^s$, there exists a whole family of nonexplosive solutions and the system is then called *indeterminate*. If $r > d^s$, the equation system is overdetermined and no nonexplosive solution exists.

Two important special case arise if all Lyapunov exponents are either positive or negative.

Corollary 2. *If all Lyapunov exponents are strictly positive, the unique solution is given by*

$$x_t = -\Phi(t,\omega)\,\mathbb{E}_t\left[\sum_{j=0}^{\infty}\Phi(t+j+1,\omega)^{-1}\,b_{t+j}\right]. \tag{3.3}$$

Proof. If all Lyapunov exponents are positive, $d^u = d$ and $\pi^u(\omega) = I_d$. In this situation, the rank condition implies that a unique solution can only arise if there is no initial condition. Hence, $x_0(\omega) = 0$ because $\pi^u(\omega)x_0(\omega) = x_0(\omega) = 0$. Clearly, any nonzero starting value would violated the bounded condition because $\Phi(t, \omega)x(\omega)$ would explode. □

Corollary 3. *If all Lyapunov exponents are strictly negative, the unique long-run solution is given by*

$$x_t = \Phi(t,\omega)\sum_{j=0}^{\infty}\Phi(t-j,\omega)^{-1}b_{t-j-1}. \tag{3.4}$$

Proof. If all Lyapunov exponents are strictly negative, $d^s = d$ and $\pi^s(\omega) = I_d$. In this situation, the rank condition determines the starting value x uniquely as the condition $\pi^u(\omega)x(\omega) = 0$ presents no restriction. Because the Lyapunov exponents are strictly negative, the term $\Phi(t, \omega)x(\omega)$ vanishes as $t \to \infty$. Hence, in the long-run only the solution (3.4) remains. □

Note that the solution (3.4) is exactly the solution derived by Brandt (1986), Bougerol and Picard (1992), and Francq and Zakoïan (2001).

4. The new Keynesian model with random policies

4.1. Specification of the random dynamics

For this approach to be of practical use, it is important to understand how the Lyapunov exponents depend on the underlying randomness and whether they are robust with regard to perturbations. In particular, one wants to know whether this dependence is continuous. While this is a delicate issue in general which is the subject of current research (see Viana (2014, chapters 9 and 10) and Backes et al. (2018)), continuity is guaranteed for the application envisaged in this paper.

More specifically, A_t is the outcome of a Markov chain with finite state space $S = \{1, 2, \dots, s\}$. To each state $j \in S$ there is associated a matrix $A^{(j)} \in \mathbb{GL}(d)$. The probability space Ω is then given by sequences of states over $\mathbb{Z}$, i.e. $\Omega = S^{\mathbb{Z}} = \{\omega \mid \omega = (\omega_j)_{j\in\mathbb{Z}}$ with $\omega_j \in S\}$, and the time shift θ is defined as $\theta\omega = \theta(\omega_j) = (\omega_{j+1})$.

The probability measure $\mathbf{P}$ on Ω is the Markov measure associated with the transition matrix P where

$$(P)_{ij} = \mathbf{P}[A(\theta\omega) = A^{(j)} \mid A(\omega) = A^{(i)}], \qquad i, j = 1, \dots, s.$$

Thereby $(P)_{ij}$ is the probability of moving to state j conditional of being in state i. We assume the Markov chain defined through P to be regular, i.e. P is irreducible (ergodic) and aperiodic. Hence, there exists a unique stationary distribution δ satisfying $\delta' P = \delta'$, $\delta \gg 0$, and $\lim_{t\to\infty} \delta_0' P^t = \delta'$ for any initial distribution δ_0. For $s = 2$, Malheiro and Viana (2015) showed that the Lyapunov exponents depend continuously on the parameters in $A^{(1)}$ and $A^{(2)}$, and P. Moreover, this is equivalent to the continuity with respect to the corresponding Lyapunov spaces (Backes and Poletti, 2017).[16] With this specification, the model becomes a Markov–switching rational expectations model which is the prime example analyzed in the economics literature so far. Hence, the results presented below are comparable to those presented there. However, it is worth emphasizing that the approach based on the MET is very versatile and can account for other $\{A_t\}$ processes.

The following example considers a Markov-switching specification with just two states where the state transition matrix is taken to be

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}, \qquad p, q \in (0, 1).$$

As p, $q \in (0, 1)$, the chain is regular with invariant distribution $\delta' = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$. Thus, δ is the unique distribution which satisfies $\delta' P = \delta'$. Hence, the chain is on average in $q/_{(p+q)}$ percent of the time in state one and $p/_{(p+q)}$ percent of the time in state two. The mean exit time from state i is $1/(1-(P)_{ii})$ which equals $1/p$ for state one and $1/q$ for state two. For $p = q = 1/2$ the chain has no memory and the sequence of states is iid. Following Shorrocks (1978), define a mobility index $M(P)$ as

$$M(P) = \frac{d - \mathrm{tr}P}{d-1} = p + q.$$

This index is equal to the reciprocal of the harmonic mean of the mean exit times.[17] It can be interpreted as measuring the randomness or mobility of the chain.

As there is in general no analytical solution for the Lyapunov exponents, numerical computations have to be applied. Because of the exponential growth of the entries in the matrix product $\Phi(t)$, numerical computations are not a straightforward task. A naive application quickly results in numerical overflows. I therefore make use of the iterative QR procedure outlined in Dieci and Elia (2008). Computations of the Lyapunov spaces are more involved and numerically sensitive (Froyland et al., 2013). The Appendix D provides more details.

4.2. Model specification

I take the canonical New Keynesian model (NK model) as my prime example. This model has been extensively analyzed in the literature and seems to be a natural first application.[18] Papers most closely related to this one are Lubik and Schorfheide (2004), Davig and Leeper (2007), Farmer et al. (2009), Galí (2011), Chen et al. (2015), Foerster (2016), and Cho (2016). A simple version of this model typically consists of the following three equations:

$$\begin{aligned}
y_t &= \mathbb{E}_t y_{t+1} - \sigma^{-1}(i_t - \mathbb{E}_t \pi_{t+1}) + u_t^d, && \text{(IS-equation)} \\
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa y_t + u_t^s, && \text{(forward-looking Phillips-curve)} \\
i_t &= \phi_t \pi_t, && \text{(Taylor-rule)}
\end{aligned}$$

[16] I thank Jairo Bochi and Anthony Quas for pointing this out to me and indicating me the relevant literature.

[17] The index is actually conceived by Shorrocks (1978) for stochastic matrices with quasi dominant diagonals. This aspect is, however, irrelevant for our purposes.

[18] See Galí (2018) for a recent assessment of the model.

Table 1
Characteristics of Transition Matrices and $\phi_{\lambda_{\min}=0}$.

specification (p, q)	invariant distribution	mean exit times			
		state 1	state 2	$M(P)$	$\phi_{\lambda_{\min}=0}$
(0.5,0.5)	(0.5, 0.5)′	2	2	1	2.43
(0.6,0.4)	(0.4, 0.6)′	5/3	5/2	1	1.73
(0.75,0.75)	(0.5, 0.5)′	4/3	4/3	1.5	2.26
(0.25,0.25)	(0.5, 0.5)′	4	4	0.5	3.86

$\phi_{\lambda_{\min}=0}$ is the cutoff value where the model switches from indeterminate to determinate

where the endogenous variables y_t, π_t, and i_t denote income (output gap), the rate of inflation and the nominal interest rate. u_t^d and u_t^s are exogenous demand and supply shocks, respectively. The parameters have the usual interpretation: $\sigma > 0$ is the intertemporal elasticity of substitution, $\beta \in (0, 1)$ is the subjective discount factor of households, and $\kappa > 0$ is the slope of the Phillips-curve. In this simplified version of the NK model there are no initial conditions so that Corollary 2 applies.

As is well-known, the determinacy of the model depends on the strength with which the monetary authority reacts to inflation. For values of ϕ below one, the model becomes indeterminate, whereas for values above one there exists a unique solution. However, monetary policy rules are not constant over time and can often, at least for some time, be characterized as policies with $0 \le \phi < 1$ (see Davig and Leeper (2007) and Galí (2011) for details and economic significance). Hence, it makes sense to treat the parameter ϕ as random and index it by t. This sensitivity of the qualitative nature of the model with respect to ϕ makes the NK model an interesting object of demonstration.

The model can be expressed in terms of $x_{t+1} = (y_{t+1}, \pi_{t+1})'$ by inserting the Taylor-rule in the IS-equation to obtain an affine random coefficient expectational difference equation of the form (2.1):

$$\mathbb{E}_t x_{t+1} = A_t x_t + b_t, \quad t \in \mathbb{Z}, \tag{4.1}$$

where

$$A_t = \frac{1}{\beta}\begin{pmatrix} \beta + (\kappa/\sigma) & (\beta\phi_t - 1)/\sigma \\ -\kappa & 1 \end{pmatrix} \quad \text{and} \quad b_t = \begin{pmatrix} u_t^d - u_t^s/(\beta\sigma) \\ u_t^s/\beta \end{pmatrix}.$$

I consider the case with two states. In the first state the central does not react to inflation at all so that $\phi_t = 0$.[19] In the second state the central bank reacts to inflation with intensity $\phi > 0$. Hence,

$$A^{(1)} = \frac{1}{\beta}\begin{pmatrix} \beta + (\kappa/\sigma) & -1/\sigma \\ -\kappa & 1 \end{pmatrix} \quad \text{and} \quad A^{(2)} = \frac{1}{\beta}\begin{pmatrix} \beta + (\kappa/\sigma) & (\phi\beta - 1)/\sigma \\ -\kappa & 1 \end{pmatrix}.$$

Because $\det A_t = \beta^{-1}(1 + \phi_t\kappa/\sigma) > 1$, $A_t \in \mathbb{GL}(2)$ irrespective of the value of ϕ_t. Both matrices are the same except for the term $(A_t)_{12}$ which is $-1/(\beta\sigma)$ in state one and $(\phi\beta - 1)/(\beta\sigma)$ in state two. I specify the values for the parameters β, κ, and σ to equal 0.985, 0.8, and 1, respectively. The value for ϕ ranges from 0 to 4 in steps of length 0.01. Finally, I examine two alternative transition matrices whose properties are summarized in Table 1.

The MET implies that there are two, not necessarily different, Lyapunov exponents $\lambda_{\max}$ and $\lambda_{\min}$, $\lambda_{\max} \ge \lambda_{\min}$. Moreover, the last assertion of the MET implies $\lambda_{\max} + \lambda_{\min} = \mathbb{E}\log|\det A(\omega)|$. As the determinant of $A(\omega)$ is always strictly greater than one, irrespective of ω, the sum of both Lyapunov exponents is always strictly greater than zero. Hence, $\lambda_{\max} > 0$. The model is therefore determinate if also $\lambda_{\min}$ is strictly greater than zero. In this case, $L^u(\omega) = \mathbb{R}^n$ and the unique solution is given by Eq. (3.3). If, however, $\lambda_{\min}$ is smaller than zero, the model becomes indeterminate. L^s and L^u are one dimensional and thus nontrivial in this case. The cutoff value ϕ where the model switches from a determinate to an indeterminate one is of special interest. This value is denoted by $\phi_{\lambda_{\min}=0}$. Because the model lacks any initial value conditions, the computation of Lyapunov spaces is unnecessary.

4.3. Simulation results

The simulation results are summarized in the bifurcation diagram in Fig. 2. Consider first the benchmark of a deterministic policy (red line). For values of ϕ below one, $\lambda_{\max} > 0 > \lambda_{\min}$ implying an indeterminate model. As the reaction of central bank to inflation increases as reflected by larger values of ϕ, $\lambda_{\min}$ increases and crosses the zero line when $\phi = 1$. For values of ϕ bigger than one, both Lyapunov exponents are positive so that the model becomes determinate with a unique solution given by Eq. (3.3). For $1 < \phi < 1.22$, the model has two distinct Lyapunov exponents corresponding to two distinct real eigenvalues. For values of ϕ greater than 1.22, the eigenvalues become conjugate complex and hence the Lyapunov exponents collapse.

[19] This case also occur if the central bank bases its policy on an inflation forecast which takes the interest path as given (Galí, 2011).

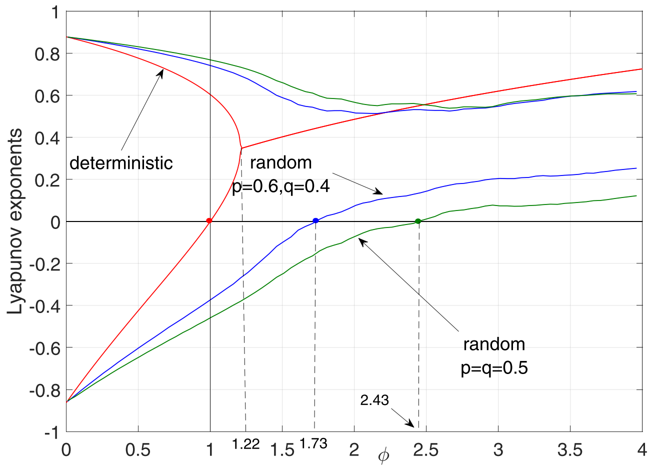

Fig. 2. Lyapunov exponents of the New Keynesian model without lagged interest rate as a function of ϕ with $\beta = 0.985$, $\kappa = 0.8$, and $\sigma = 1$.

When the policy is no longer fixed, but random, a qualitatively similar picture emerges. Consider first the case $p = q = 0.5$ (green line) so that the system alternates in an iid fashion between the two states with mean exit time from each state equal to two periods. Hence, the central bank reacts on average only in 50 percent of the time to inflation. As shown in Fig. 2, there are two distinct Lyapunov exponents. Because the economy spends some time in state one with no reaction to inflation, the central bank must react more strongly in the state where it takes inflation into account. According to our simulation result, it must react with an intensity ϕ greater than $\phi_{\lambda_{\min}=0} = 2.43$ in state two to obtain a determinate model. When the central bank follows an active anti-inflationary policy more often as reflected by the specification $p = 0.6, q = 0.4$, the reaction to inflation in state two can be less strong. The model already becomes determinate for values of ϕ bigger than $\phi_{\lambda_{\min}=0} = 1.73$ (blue line). This exercise clearly delineates a trade-off between a strong reaction to inflation in state two and the time spent in state two.

In the previous simulations the mobility index was the same. In this sense both Markov chains analyzed previously exhibit the same volatility. When the chain becomes more volatile as in case 3, the mean exit times are reduced from 2 to 4/3 and $\phi_{\lambda_{\min}=0}$ equals 2.26 which is lower than in case 1. Note that case 1 and 3 have the same invariant distribution. If the mobility is reduced so that the chain stays on average longer in one state, the reaction in the active state must be much more aggressive. As reported in Table 1 the cutoff value $\phi_{\lambda_{\min}=0}$ increases to 3.86.

In a next step I generalize the New Keynesian model above by allowing for lagged endogenous variables. Following Cho (2016), I include a lagged interest rate and a monetary disturbance u_t^m in the Taylor rule:

$$i_t = (1-\rho)\phi\pi_t + \rho i_{t-1} + u_t^m, \qquad 0 < \rho < 1.$$

This generalization can be easily accommodated within the theoretical framework presented before by enlarging the state vector by i_t. Hence, $x_{t+1} = (y_{t+1}, \pi_{t+1}, i_t)'$ and the system matrices are given as

$$A_t = \begin{pmatrix} 1 & -(\beta\sigma)^{-1} & \sigma^{-1} \\ 0 & \beta^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\kappa & 1 & 0 \\ 0 & (1-\rho)\phi_t & \rho \end{pmatrix}$$

and

$$b_t = \begin{pmatrix} -1 & (\beta\sigma)^{-1} & \sigma^{-1} \\ 0 & -\beta^{-1} & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} u_t^d \\ u_t^s \\ u_t^m \end{pmatrix}$$

Note that $A_t \in \mathbb{GL}(3)$, irrespective of the value of ϕ_t, because $\det A_t = \rho/\beta > 0$. Because of this increase in the dimension of the state vector, the number of Lyapunov exponents rises to 3. As the value of i_t is given and known in period t, the modified model has effectively one initial condition. Hence, Corollary 1 implies that a determinate model requires one negative and two positive Lyapunov exponents.

Leaving the values of all parameters as before and setting $\rho = 0.7$ produces the results summarized in Fig. 3. In this figure I have for comparison purposes omitted to display the third Lyapunov exponent. This exponent is always negative and delivers no additional information concerning the determinateness of the model. As one can deduce from Fig. 3, the inclusion of a lagged interest rate in the Taylor rule does not alter the qualitative features of the model.

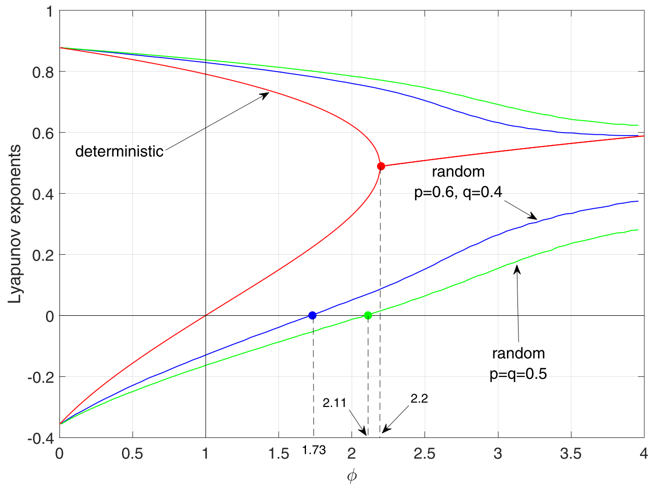

Fig. 3. Lyapunov exponents of the New Keynesian model with lagged interest rate as a function of ϕ with $\beta = 0.985$, $\kappa = 0.8$, $\sigma = 1$, and $\rho = 0.7$.

5. Conclusion

The purpose of this paper was to present to economists the mathematical tools which would enable them to analyze rational expectations models with time-varying coefficients. The methodology is rather versatile and is not restricted to Markov-switching models which is the predominant approach so far. As long as the process generating the time variation in the coefficients is strictly stationary and ergodic the approach outlined in this paper can be applied. This includes in particular models with predetermined or lagged endogenous variables.

The theoretical core of this methodology evolves around the concept of Lyapunov exponents which measure the asymptotic growth rates of trajectories. The Multiplicative Ergodic Theorem by Oseledets then showed that the Lyapunov exponents play a similar role in the analysis of the stability of random dynamical systems as the eigenvalues do in the standard case of constant coefficients. Based in this insight, the paper constructs an explicit solution formula which typically involve backwards as well as forward looking components. The latter can be interpreted in terms of “present discounted values” and are thus akin to economics. This aspect brings the paper close to the spirit of the standard Blanchard–Kahn analysis of rational expectations models with constant coefficients. The methodology is also relevant for the analysis of regime–switching time series models á la Hamilton (1989, 2016). In this literature, however, the emphasis is on the top Lyapunov exponent whose negativity guarantees the existence of a stationary solution (see Bougerol and Picard, 1992; Brandt, 1986; Francq and Zakoïan, 2001).

The application of these tools requires numerical methods as analytical solutions are almost never available. Fortunately, powerful procedures to estimate the Lyapunov exponents as well as the corresponding Lyapunov spaces have been developed (see Dieci and Elia (2008) and Froyland et al. (2013), f.e.). Finally, the paper runs a simulation exercise of a prototype New Keynesian model with random Taylor rule and a lagged endogenous variable to demonstrate the practical usefulness of the approach. This exercise demonstrated that there are no conceptual obstacles to apply this methodology to more sophisticated models.

Appendix A. Lyapunov Exponents and Eigenvalues

For a better understanding of the arguments presented in this paper, it is instructive to examine the case of constant coefficients. Thus, $x_t = \varphi(t, \omega, x) = \varphi(t, x) = A^t x$ and the Lyapunov exponents are just the logarithms of the distinct moduli of the eigenvalues of A (Colonius and Kliemann, 2014, section 1.5). Denote the different Lyapunov exponents by $\lambda_1 > \ldots > \lambda_\ell$. Then, to each Lyapunov exponent λ_j there is associated a subspace L_j, called *Lyapunov space*, defined as $L_j = \bigoplus E_k$ where the direct sum is taken over all real generalized eigenspaces E_k related to eigenvalues μ_k with $\lambda_j = \log|\mu_k|$. The state space $\mathbb{R}^d$ can then be decomposed into a direct sum of these Lyapunov spaces:

$$\mathbb{R}^d = L_1 \oplus \cdots \oplus L_\ell.$$

The definition of eigenvalues and eigenvectors imply that the Lyapunov spaces are invariant with respect to A, i.e. $AL_j = L_j$ for $j = 1, \ldots, \ell$. This property is called *equivariance*.

Moreover, for any solution $\varphi(t, x)$ with $x \neq 0$

$$\lambda(x) = \lim_{t\to\pm\infty} \frac{1}{t} \log\|\varphi(t, x)\| = \lambda_j \text{ if and only if } x \in L_j.$$

The above characterization of Lyapunov exponents and Lyapunov spaces requires to take the double-sided limit. To see this, suppose, for the sake of the argument, that $d=2$, $A=\operatorname{diag}(\mu_1,\mu_2)$ with $\mu_1 \neq \mu_2 \in \mathbb{R}$ and $|\mu_1|>|\mu_2|$. Furthermore, let $\|.\|$ be the max–norm and denote $x=(x_1,x_2)'$. The Lyapunov exponents are therefore computed as

$$\lambda(x) = \limsup \frac{1}{t}\log\max\{|\mu_1^t x_1|, |\mu_2^t x_2|\}.$$

In this case, the Lyapunov exponents are $\lambda_1=\log|\mu_1|$ and $\lambda_2=\log|\mu_2|$ with the corresponding Lyapunov spaces $L(\lambda_1)=\operatorname{span}(1,0)'$ and $L(\lambda_2)=\operatorname{span}(0,1)'$. The above characterization then reads

$$\lambda(x) = \lim_{t\pm\infty} \frac{1}{t}\log\max\{|\mu_1^t x_1|, |\mu_2^t x_2|\} = \lambda_1 \Leftrightarrow x \in L(\lambda_1) = \operatorname{span}(1,0)'.$$

Thus, the if-and-only-if statement only holds for the two-sided limit because for $t \to -\infty$ the above expression is dominated by $|\mu_2|^t$. The same argument applies for λ_2 and $L(\lambda_2)$.

In order to prepare for the random coefficient case, it is instructive to invoke the *spectral theorem* (Meyer, 2000, chapters 7.2 and 7.3). let A be diagonalizable with spectrum $\sigma(A)=\{\mu_1,\dots,\mu_\ell\}$, then

$$A^t x = \mu_1^t \pi_1 x + \cdots + \mu_\ell^t \pi_\ell x$$

where π_j denote the projectors onto $\mathbf{N}(A-\mu_j I_d)$ along $\mathbf{R}(A-\mu_j I_d)$, $j=1,\dots,\ell$. They have the properties that $\pi_i\pi_j=0$ whenever $i \neq j$ and that $\pi_1+\dots\pi_\ell = I_d$. Hence, for an arbitrary $x\in\mathbb{R}^d$ the asymptotic behavior is dominated by the largest eigenvalue in absolute terms, say μ_1. If, however, x is chosen specifically as an element of the complement of $\mathbf{N}(A-\mu_1 I_d)$, i.e. such that $\pi_1 x = 0$, the asymptotic behavior is governed by the second largest eigenvalue in absolute terms. Obviously, one proceed further in this manner until all eigenvalues are exhausted. From the diagonal representation of A, the projections π_j are given by

$$\pi_j = Q\begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{d_j} & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^{-1}, \qquad j=1,\dots,\ell$$

where d_j is the dimension of the eigenspace related to eigenvalue μ_j and where the columns of Q consist of generalized eigenvectors which form a basis of $\mathbb{R}^d$.

Appendix B. More on the MET

Relation to Birkhoff's ergodic theorem To justify the labeling *ergodic* in the MET, I relate it to Birkhoff's ergodic theorem. Let the state space be one-dimensional, i.e. $d=1$. Then define $f(\omega)=\log|a(\omega)|$ where $a(\omega)$ stands for $A(\omega)$. The ergodicity of θ and the integrability condition imply that the prerequisites of Birkhoff's pointwise ergodic theorem are satisfied.[20] Thus,

$$\frac{1}{t}\sum_{j=0}^{t-1} f(\theta^j\omega) \longrightarrow \int_\Omega f\mathrm{d}\mathbf{P} = \mathbb{E}\log|a(\omega)| = \lambda.$$

As $\log|\varphi(t,\omega,x)| = \sum_{j=0}^{t-1}\log|a(\theta^j\omega)| + \log|x|$, statement *(iii)* in the MET corresponds exactly the Birkoff's theorem because $\lim_{t\to\pm\infty}\frac{1}{t}|x|$ converges to zero. Birkhoff's theorem, however, cannot be immediately generalized to higher dimensions because the matrix multiplication is not commutative.

The case of triangular matrices To get a more profound understanding of the MET, it is instructive to examine the case of triangular matrices.[21] Let $A(\omega)\in\mathbb{GL}(2)$ be upper triangular:

$$A(\omega) = \begin{pmatrix} a(\omega) & c(\omega) \\ 0 & b(\omega) \end{pmatrix}, \qquad a(\omega)\neq 0 \text{ and } b(\omega)\neq 0.$$

$\Phi(t)$ is then given by

$$\Phi(t) = A_{t-1}\dots A_1 A_0 = \begin{pmatrix} \prod_{j=0}^{t-1} a_j & \sum_{k=0}^{t-1} a_{t-1}\dots a_{k+1}\, c_k\, b_{k-1}\dots b_0 \\ 0 & \prod_{j=0}^{t-1} b_j \end{pmatrix}.$$

[20] An introduction to ergodic theory can be found, f.e., Silva (2008, chapter 5), Grimmett and Stirzaker (2001, section 9.5), or Colonius and Kliemann (2014, section 10.1).

[21] This exposition is based on Arnold (2003, p.129–130) and Berger (1993, p.155).

Note that $\mathbb{R}e_1 = \text{span}(1, 0)'$ is an invariant subspace for the $\Phi(t)$'s. Let the stochastic sequences $\{a_t\}$, $\{b_t\}$, and $\{c_t\}$ be ergodic with $\alpha = \mathbb{E}\log|a_t|$, $\beta = \mathbb{E}\log|b_t|$, and $\gamma = \mathbb{E}\log|c_t|$. Then

$$\frac{1}{t}\sum_{j=0}^{\infty}\log|a_t| \to \alpha \quad \text{and} \quad \frac{1}{t}\sum_{j=0}^{\infty}\log|b_t| \to \beta.$$

Hence,

$$\frac{1}{t}\log|\det\Phi(t)| \to \alpha+\beta.$$

From implication *(v)* of the MET, it follows that $\lambda_1 + \lambda_2 = \alpha + \beta$. Obviously, the Lyapunov exponents of $[\Phi(t, \omega)]_{11}$ and $[\Phi(t, \omega)]_{22}$ are α and β, respectively. Moreover, the Lyapunov exponent of $[\Phi(t, \omega)]_{12}$ is less than or equal to $\max\{\alpha, \beta\}$. The subadditivity of the lim sup implies $\lambda(x+y) \le \max\{\lambda(x), \lambda(y)\}$, with equality if $\lambda(x) \neq \lambda(y)$, hence

$$\frac{1}{t}\log\|\Phi(t,\omega)\| \to \lambda_1 = \max\{\alpha,\beta\}.$$

Thus,

$$\lambda_1 = \max\{\alpha,\beta\} > \frac{\alpha+\beta}{2} > \lambda_2 = \min\{\alpha,\beta\} \quad \text{for } \alpha \neq \beta.$$

When $\alpha = \beta$, $\lambda_1 = \alpha = \beta$ with multiplicity 2.

Because $A(\omega)$ are triangular matrices, $\Phi(t, \omega)$ as a product of triangular matrices is also triangular with its eigenvalues $\mu_1(t, \omega)$ and $\mu_2(t, \omega)$ sitting on the diagonal. Hence, $\mu_1(t,\omega) = \prod_{j=0}^{t-1} a_j$ and $\mu_2(t,\omega) = \prod_{j=0}^{t-1} b_j$. Setting $|\mu_i(t,\omega)| \approx e^{t\lambda_i}$, the Lyapunov exponents are nothing but the mean exponential rate of growth of $|\mu_i(t, \omega)|$ as $t \to \infty$:

$$\lambda_i = \lim_{t\to\infty}\frac{1}{t}\log|\mu_i(t,\omega)|, \qquad i = 1, 2.$$

Although the eigenvalues are stochastic, the Lyapunov exponents are not due to the ergodicity of $\{a_t\}$, $\{b_t\}$, and $\{c_t\}$. The corresponding eigenspaces, however, will remain stochastic as the computation below shows. This interpretation of Lyapunov exponents\spaces remains valid also in the case of non-triangular matrices due to the iterative QR decomposition exposed in Appendix D.

To compute the Lyapunov spaces, assume without loss of generality $\alpha > \beta$, hence $\lambda_1 = \alpha$ and $\lambda_2 = \beta$. For any vector $x = (x_1, 1)'$ to grow at rate β, x must be an eigenvector with respect to eigenvalue $b(t) = [\Phi(t)]_{22}$. Thus

$$\begin{pmatrix} a(t) & c(t) \\ 0 & b(t) \end{pmatrix}\begin{pmatrix} x_1 \\ 1 \end{pmatrix} = b(t)\begin{pmatrix} x_1 \\ 1 \end{pmatrix}$$

with $a(t) = [\Phi(t)]_{11}$ and $c(t) = [\Phi(t)]_{12}$. Taking limits and recognizing that $b(t)/a(t) \to 0$, one obtains

$$x_1 = -\lim_{t\to\infty}\frac{c(t)}{a(t)} = -\sum_{k=0}^{\infty}\frac{c_k b_{k-1}\dots b_1 b_0}{a_k\dots a_1 a_0}.$$

This defines the random Lyapunov subspace $L(\lambda_2) = L(\beta) = \text{span}(x_1, 1)'$. Because x has to grow at rate β, x must be random. Moreover, this randomness depends on the entire sequence $\{A_t\}$. The other Lyapunov subspace is, as noted before, $L(\lambda_1) = L(\alpha) = \mathbb{R}e_1$.

Appendix C. Proof of Theorem 2

The first part of the proof replicates the proof proposed by Arnold and Crauel (1992) and Arnold (2003, theorem 5.6.5) omitting some of technical details.

Proof. First I prove that the expressions for $x_t^{(b)}$ and $x_t^{(f)}$ are well-defined. Consider for this purpose a trajectory starting in x denoted by $\varphi(t, \omega, x)$. Iterating the difference equation backward

$$\varphi(t,\omega,x) = \Phi(t,\omega)x + \Phi(t,\omega)\sum_{j=0}^{t-1}\Phi(t-j,\omega)^{-1}b(\theta^{t-1-j}\omega).$$

This implies

$$\begin{aligned}\pi^s(\theta^t\omega)\varphi(t,\omega,x) &= \pi^s(\theta^t\omega)\Phi(t,\omega)x \\ &\quad + \pi^s(\theta^t\omega)\Phi(t,\omega)\sum_{j=0}^{t-1}\Phi(t-j,\omega)^{-1}b(\theta^{t-1-j}\omega) \\ &= \Phi(t,\omega)\pi^s(\omega)x \\ &\quad + \Phi(t,\omega)\sum_{j=0}^{t-1}\Phi(t-j,\omega)^{-1}\pi^s(\theta^{t-j}\omega)b(\theta^{t-1-j}\omega)\end{aligned} \tag{C.1}$$

Next choose $\beta \in (0, \min\{-\lambda_s - \kappa, \lambda_u + \kappa\}$ where κ satisfies $\lambda_s = \max_{\lambda_j<0} \lambda_j < -\kappa < 0 < \kappa < \lambda_u = \min_{\lambda_j>0} \lambda_j$. The first term in the above expression then converges exponentially fast to zero because $\left\| \Phi(t,\omega)|_{L^s(\omega)} \right\| \le e^{-\beta t}$.

The integrability condition $\mathbb{E}\log^+ \|b\| < \infty$ implies

$$\sum_{j=0}^{\infty} \mathbf{P}\left[\log^+ \|b(\theta^{t-j-1}\omega)\| > \frac{\beta}{2}\right] < \infty.$$

This implies by the Borel–Cantelli lemma that almost surely

$$\limsup_{j\ge 0} \frac{1}{j} \log^+ \|b(\theta^{t-j-1}\omega)\| \le \frac{\beta}{2}.$$

Hence

$$\limsup_{j\ge 0} \frac{1}{j} \log \|\Phi(j,\theta^{t-j}\omega)\pi^s(\theta^{t-j}\omega)b(\theta^{t-1-j}\omega)\| \le \limsup_{j\ge 0} \frac{1}{j} \log \|\Phi(j,\theta^{t-j}\omega)\| \, \|\pi^s(\theta^{t-j}\omega)b(\theta^{t-1-j}\omega)\| \le \frac{-\beta}{2}.$$

The second term in Eq. (C.1) therefore is

$$\Phi(t,\omega)\sum_{j=0}^{t-1} \Phi(t-j,\omega)^{-1}\pi^s(\theta^{t-j}\omega)b(\theta^{t-1-j}\omega) = \sum_{j=0}^{t-1} \Phi(j,\theta^{t-j}\omega)^{-1}\pi^s(\theta^{t-j}\omega)b(\theta^{t-1-j}\omega)$$

converges almost surely to $x_t^{(b)}$. Thus, $x_t^{(b)}$ is well-defined. A similar argument can be made with respect $x_t^{(f)}$

Using the towering property of conditional expectations and omitting the dependence on ω whenever possible, the solution given in Eq. (3.1) indeed solves the expectational difference equation:

$$\begin{aligned}
\mathbb{E}_t x_{t+1} = \mathbb{E}_t \Bigg\{ & \Phi(t+1)x + \Phi(t+1)\sum_{j=0}^{\infty} \Phi(t+1-j)^{-1}\pi^s(\theta^{t+1-j}\omega)b_{t-j} \\
& - \Phi(t+1)\mathbb{E}_{t+1}\left[\sum_{j=0}^{\infty} \Phi(t+j+2)^{-1}\pi^u(\theta^{t+j+2}\omega)b_{t+1+j}\right]\Bigg\} \\
= {} & A_t\Phi(t)x + \Phi(t+1)\Phi(t+1)^{-1}\pi^s(\theta^{t+1}\omega)b_t \\
& + A_t\Phi(t)\sum_{j=0}^{\infty} \Phi(t-j)^{-1}\pi^s(\theta^{t-j}\omega)b_{t-1-j} \\
& - A_t\Phi(t)\mathbb{E}_t\left[\sum_{j=0}^{\infty} \Phi(t+j+1)\pi^u(\theta^{t+j+1}\omega)b_{t+j}\right] \\
& + \Phi(t+1)\Phi(t+1)^{-1}\pi^u(\theta^{t+1}\omega)b_t \\
= {} & A_t x_t + (\pi^s(\theta^{t+1}\omega) + \pi^u(\theta^{t+1}\omega))b_t = A_t x_t + b_t.
\end{aligned}$$

The solution (3.1) depends parametrically on the unknown $x(\omega)$. In order to fix $x(\omega)$, I resort to the initial value condition $Rx_0(\omega) = c$ which implies $c - Rx_0^{(p)}(\omega) = Rx(\omega)$. The boundedness condition further requires that $x(\omega) \in L^s(\omega)$ or, equivalently, that $\pi^u(\omega)x(\omega) = 0$. These two conditions then determine $x(\omega)$ uniquely if the rank condition (3.2) is satisfied. □

Appendix D. Computing Lyapunov Exponents

Although Lyapunov exponents and their corresponding subspaces are straightforwardly defined, their computation presents some numerical pitfalls. The reason stems from the exponential growth of the entries in the matrix product $\Phi(t)$, respectively Υ, as t becomes large. Trying to compute these matrices directly, very quickly leads to numerical overflows on any computer. One way to circumvent this problem is to rescale and reorthogonalize the matrix at each step. The iterative QR and SVD decompositions are two such prominent algorithms. Their stability and numerical accuracy have been investigated by Stewart (1997, 1995), and particularly relevant for this paper by Dieci and Elia (2008). These analyses suggest that the methodologies proposed therein gives satisfactory results even for relatively large models (i.e. models with dimensions up to 100).

In this paper, I rely on the QR approach which is easy to implement.[22] In the case of a Markov-switching model corresponding to the example in Section 4, the algorithm proceeds as follows.

Initialization Take $X(0)$ equal to any feasible matrix A_0 chosen randomly from the set $\{A^{(j)}, j \in S\}$ according to the invariant distribution δ. Compute the QR decomposition $\Phi(1) = X(0) = Q_0 R_0$.

[22] When the matrices are nearly singular the QR approach may be improved by column pivoting.

Iteration Given a QR decomposition of $\Phi(k) = Q_{k-1}R_{k-1}\dots R_1R_0$, take a draw A_k according to the Markov chain governing the evolution of $\{A_t\}$. Then compute $X(k) = A_kX(k-1)$ and perform the QR decomposition of $X(k)Q_{k-1} = Q_kR_k$. $\Phi(k+1)$ is then obtained as

$$\Phi(k+1) = X(k)\Phi(k) = X(k)Q_{k-1}Q'_{k-1}\Phi(k) = Q_kR_k\dots R_1R_0.$$

Lyapunov exponents Generalizing the arguments made in Appendix B for triangular 2×2 matrices to $d\times d$ triangular matrices, approximate Lyapunov exponents $\lambda_j^{(k+1)}$ can then be computed as

$$\lambda_j^{(k+1)} = \frac{1}{k+1}\log\prod_{n=0}^{k}[R_n]_{jj} = \frac{1}{k+1}\sum_{n=0}^{k}\log[R_n]_{jj}, \quad j = 1,\dots,d,$$

where $[R_n]_{jj}$ denotes the j-th diagonal element of R_n.

Stop criterion Continue the iteration until a sufficient precision for the Lyapunov exponents is achieved.

This procedure effectively produces a QR decomposition of $\Phi(t)$;

$$\Phi(t) = \underbrace{Q_{t-1}}_{\text{orthogonal}}\ \underbrace{R_{t-1}\dots R_1R_0}_{\text{triangular}}$$

because the product of triangular matrices is triangular. Moreover, because of the orthogonality of Q_{t-1}, $\|\Phi(t)\| = \|Q_{t-1}R_{t-1}\dots R_1R_0\| = \|R_{t-1}\dots R_1R_0\|$. Hence the matrix Q_k can be disregarded when computing the approximate Lyapunov exponents.[23] For further details and improved algorithms for higher dimensional models can be found in Dieci and Elia (2008) and the literature cited therein.

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[23] MATLAB codes for the examples in the paper are available on the author's homepage. It took about 0.06 seconds, respectively about 2600 iterations, to compute the Lyapunov exponents with a precision of four digits on a Surface notebook (Intel Core i7-6650U CPU with 2.20 GHz and 16 GB RAM). The computation for the model with a lagged interest rate in the Taylor rule was even faster.

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